Decomposition techniques for optimization problems with complicating variables

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Learning Objectives

After this session the participants are expected to be able to:

• Explain the functioning of Benders’ decomposition
• Explain the application of Benders’ decomposition to multi-stage stochastic problems
Objective function (total cost)

Complicating variable

Potential problems:
- Investment decisions
- Productions schedules among different markets
- etc

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Concept

Objective function
(total cost)

Complicating variable

Upper bound
Optimal value for complicating variable is trivial, if the cost profile is available!
But what if it is not?

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Assume the cost profile is not available, how to determine the optimal point by mathematical tricks?
Step 1) Choose arbitrarily a value for complicating variable
Concept

Step 2) Calculate the corresponding i) cost, and ii) sensitivity of cost with respect to selected complicating variable

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**Concept**

- **Objective function** (total cost)

**Complicating variable**

**Step 3)** Draw a constraint (cut) using the cost and sensitivity (slope) obtained.

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Step 4) Simulate the approximate cost profile using the cut obtained
Step 5) Determine the optimal point as the next iteration
Step 6) Draw the new cut, and simulate the updated cost profile.

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Concept

Objective function (total cost)

Complicating variable

Next steps) new cuts, and more accurate approximate cost profile!
Two-stage Deterministic Problem

\[
\begin{align*}
\min_{x_1, x_2} \quad & c_1 x_1 + c_2 x_2 \\
\text{subject to} \quad & A_1 x_1 \geq b_1 \\
& E_1 x_1 + A_2 x_2 \geq b_2
\end{align*}
\]
Two-stage Deterministic Problem

\[
\begin{align*}
\min_{x_1, x_2} & \quad c_1 x_1 + c_2 x_2 \\
\text{subject to} & \quad A_1 x_1 \geq b_1 \\
& \quad E_1 x_1 + A_2 x_2 \geq b_2
\end{align*}
\]

First-stage cost

Second-stage cost

First-stage constraint

Linking constraint

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Two-stage Deterministic Problem

First-stage problem:

$$\min_{x_1} \ c_1 x_1 + \alpha_1(x_1)$$
subject to
$$A_1 x_1 \geq b_1$$

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \ c_2 x_2$$
subject to
$$A_2 x_2 \geq b_2 - E_1 x_1$$

$$\alpha_1(x_1)$$: the second-stage cost as a function of the first-stage decisions $$x_1$$ (future cost function)

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Two-stage Deterministic Problem

First-stage problem:

\[
\min_{x_1} \quad c_1 x_1 + \alpha_1(x_1)
\]

subject to

\[
A_1 x_1 \geq b_1
\]

Second-stage problem:

\[
\alpha_1(x_1) = \min_{x_2} \quad c_2 x_2
\]

subject to

\[
A_2 x_2 \geq b_2 - E_1 x_1
\]

\(\alpha_1(x_1)\): the second-stage cost as a function of the first-stage decisions \(x_1\)
(future cost function)

Note: \(x_1\) appears in the second-stage problem!

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Two-stage Deterministic Problem

First-stage problem:

\[
\min_{x_1} \ c_1 x_1 + \alpha_1(x_1)
\]

subject to

\[
A_1 x_1 \geq b_1
\]

Second-stage problem:

\[
\alpha_1(x_1) = \min_{x_2} \ c_2 x_2
\]

subject to

\[
A_2 x_2 \geq b_2 - E_1 x_1
\]
Two-stage Deterministic Problem

One potential solution approach

- **Step 1)** **Discretize** $x_1$ into a set of trial values $\{\hat{x}_{1i}, i = 1, \ldots, n\}$

- **Step 2)** Solve the second-stage problem for each of the trial values

- **Step 3)** Construct future cost function $\alpha_1(x_1)$. Intermediate values of $\alpha_1(x_1)$ are obtained by interpolation from the neighboring discretized values.

- **Step 4)** Solve the first-stage problem using the future cost function constructed.

**First-stage problem:**

$$\min_{x_1} c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

**Second-stage problem:**

$$\alpha_1(x_1) = \min_{x_2} c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1$$
Two-stage Deterministic Problem

One potential solution approach

- **Step 1)** Discretize $x_1$ into a set of trial values $\{\hat{x}_{1i}, i = 1, ..., n\}$
- **Step 2)** Solve the second-stage problem for each of the trial values
- **Step 3)** Construct future cost function $\alpha_1(x_1)$. Intermediate values of $\alpha_1(x_1)$ are obtained by interpolation from the neighboring discretized values.
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Two-stage Deterministic Problem

One potential solution approach

- **Step 1)** Discretize $x_1$ into a set of trial values $\{\hat{x}_{1i}, i = 1, ..., n\}$
- **Step 2)** Solve the second-stage problem for each of the trial values
- **Step 3)** Construct future cost function $\alpha_1(x_1)$. Intermediate values of $\alpha_1(x_1)$ are obtained by interpolation from the neighboring discretized values.
- **Step 4)** Solve the first-stage problem using the future cost function constructed.

What is the name of this technique?
Two-stage Deterministic Problem

One potential solution approach

- **Step 1)** Discretize $x_1$ into a set of trial values $\{\hat{x}_{1i}, i = 1, ..., n\}$
- **Step 2)** Solve the second-stage problem for each of the trial values
- **Step 3)** Construct future cost function $\alpha_1(x_1)$. Intermediate values of $\alpha_1(x_1)$ are obtained by interpolation from the neighboring discretized values.
- **Step 4)** Solve the first-stage problem using the future cost function constructed.

What is the name of this technique?

**Dynamic programming (DP)**

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Two-stage Deterministic Problem

One potential solution approach

What is the main drawback of dynamic programming (DP)?
Two-stage Deterministic Problem

One potential solution approach

What is the main drawback of dynamic programming (DP)?

DP needs to discretize the decision variables $x_1$, which results in computational issues!

For example, 10 decision variables and 4 discretized value for each variable leads to $4^{10}$ discrete values!
Two-stage Deterministic Problem

*Alternative solution approach*
Two-stage Deterministic Problem

Alternative solution approach

**Dual** dynamic programming (DDP) instead of DP!

**Advantage:**
To approximate the future cost function $\alpha_1(x_1)$ by analytical functions rather than a set of discrete values!
Two-stage Deterministic Problem

DDP functioning procedure

\[ \alpha_1(x_1) = \min_{x_2} c_2 x_2 \]

subject to

\[ A_2 x_2 \geq b_2 - E_1 x_1 : \pi \]
Two-stage Deterministic Problem

$\alpha_1(x_1) = \min_{x_2} c_2 x_2$

subject to

$A_2 x_2 \geq b_2 - E_1 x_1 : \pi$

Dual of the second-stage problem:

$max_{\pi} \pi(b_2 - E_1 x_1)$

subject to

$\pi A_2 \geq c_2$
Two-stage Deterministic Problem

DDP functioning procedure

\[
\alpha_1(x_1) = \min_{x_2} c_2 x_2
\]
subject to
\[
A_2 x_2 \geq b_2 - E_1 x_1 : \pi
\]

Dual of the second-stage problem:

\[
\max_{\pi} \pi (b_2 - E_1 x_1)
\]
subject to
\[
\pi A_2 \geq c_2
\]

In the optimal solution
(strong duality theorem)

\[
\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)
\]
Two-stage Deterministic Problem

**DDP functioning procedure**

\[
\alpha_1(x_1) = \min_{x_2} c_2x_2
\]
subject to
\[
A_2x_2 \geq b_2 - E_1x_1 : \pi
\]

**Interpretation:** there is a linear relation between \(x_1\) and the future cost function \(\alpha_1(x_1)\) if the sensitivity \(\pi^*\) is known!

**Dual of the second-stage problem:**

\[
\max_{\pi} \pi(b_2 - E_1x_1)
\]
subject to
\[
\pi A_2 \geq c_2
\]

\[
\alpha_1(x_1) = \pi^*(b_2 - E_1x_1)
\]

**In the optimal solution**

*(strong duality theorem)*
Two-stage Deterministic Problem

DDP functioning procedure

\[ \alpha_1(x_1) = \pi^*(b_2 - E_1x_1) \]
Two-stage Deterministic Problem

**DDP functioning procedure**

\[ \alpha_1(x_1) = \pi^*(b_2 - E_1 x_1) \]

Assume \( \pi^1, \pi^2, \ldots, \pi^n \) are possible values for \( \pi^* \). Then, \( \alpha_1(x_1) \) can be characterized as follows:

\[
\begin{align*}
\alpha_1(x_1) = & \min_{\alpha, x_1} \alpha \\
& \text{subject to} \\
& \alpha \geq \pi^1(b_2 - E_1 x_1) \\
& \alpha \geq \pi^2(b_2 - E_1 x_1) \\
& \quad \ldots \\
& \alpha \geq \pi^n(b_2 - E_1 x_1)
\end{align*}
\]
Two-stage Deterministic Problem

DDP functioning procedure

\[ \alpha_1(x_1) = \pi^*(b_2 - E_1x_1) \]

Assume \( \pi^1, \pi^2, ..., \pi^n \) are possible values for \( \pi^* \). Then, \( \alpha_1(x_1) \) can be characterized as follows:

\[ \alpha_1(x_1) = \min_{\alpha, x_1} \alpha \]

subject to

\[ \alpha \geq \pi^1(b_2 - E_1x_1) \]
\[ \alpha \geq \pi^2(b_2 - E_1x_1) \]
\[ \vdots \]
\[ \alpha \geq \pi^n(b_2 - E_1x_1) \]

Let’s interpret this optimization problem!
Two-stage Deterministic Problem

**DDP functioning procedure**

\[ \alpha_1(x_1) = \pi^*(b_2 - E_1x_1) \]

Assume \( \pi^1, \pi^2, ..., \pi^n \) are possible values for \( \pi^* \). Then, \( \alpha_1(x_1) \) can be characterized as follows:

\[
\begin{align*}
\alpha_1(x_1) &= \min_{\alpha, x_1} \alpha \\
\text{subject to} \\
\alpha &\geq \pi^1(b_2 - E_1x_1) \\
\alpha &\geq \pi^2(b_2 - E_1x_1) \\
&\quad \vdots \\
\alpha &\geq \pi^n(b_2 - E_1x_1)
\end{align*}
\]

**Note:** this means that we can construct a piecewise linear function for \( \alpha_1(x_1) \) problem (analytically but approximately) without need to discretize \( x_1 \)!
Two-stage Deterministic Problem

*DDP functioning procedure*

\[ \alpha_1(x_1) = \pi^*(b_2 - E_1x_1) \]

Assume \( \pi^1, \pi^2, \ldots, \pi^n \) are possible values for \( \pi^* \). Then, \( \alpha_1(x_1) \) can be characterized as follows:

\[
\begin{align*}
\alpha_1(x_1) &= \min_{\alpha, x_1} \alpha \\
&\text{subject to} \\
\alpha &\geq \pi^1(b_2 - E_1x_1) \\
\alpha &\geq \pi^2(b_2 - E_1x_1) \\
&\quad \vdots \\
\alpha &\geq \pi^n(b_2 - E_1x_1)
\end{align*}
\]

Recall the first-stage problem:

\[
\begin{align*}
\min_{x_1} \quad &c_1x_1 + \alpha_1(x_1) \\
&\text{subject to} \\
&\ A_1x_1 \geq b_1
\end{align*}
\]
Two-stage Deterministic Problem

Assume $\pi^1, \pi^2, ..., \pi^n$ are possible values for $\pi^*$. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$
\alpha_1(x_1) = \min_{\alpha, x_1} \alpha
$$

subject to

$$
\alpha \geq \pi^1(b_2 - E_1x_1)
$$

$$
\alpha \geq \pi^2(b_2 - E_1x_1)
$$

. . .

$$
\alpha \geq \pi^n(b_2 - E_1x_1)
$$

Recall the first-stage problem:

$$
\min_{x_1} c_1x_1 + \alpha_1(x_1)
$$

subject to

$$
A_1x_1 \geq b_1
$$

Let’s merge them!
Two-stage Deterministic Problem

DDP functioning procedure

Updated first-stage problem including the piecewise linear function $\alpha_1(x_1)$

$$\min_{\alpha, x_1} c_1 x_1 + \alpha$$

subject to

$$A_1 x_1 \geq b_1$$

$$\alpha \geq \pi^1 (b_2 - E_1 x_1)$$

$$\alpha \geq \pi^2 (b_2 - E_1 x_1)$$

.$$\alpha \geq \pi^n (b_2 - E_1 x_1)$$

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How to generate possible values for $\pi^*$, i.e., $\pi^1, \pi^2, ..., \pi^n$?
Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for $\pi^*$, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 1:

1. Generate trial values for $x_1$
   
   $$(\hat{x}_{1i}, i = 1, \ldots, n)$$

2. Solve the second-stage problem for each trial, and obtain $\pi^1, \pi^2, \ldots, \pi^n$

3. Solve the first-stage problem
Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for $\pi^*$, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 1:

Generate trial values for $x_1$

$(\hat{x}_{1i}, i = 1, \ldots, n)$

Solve the second-stage problem for each trial, and obtain $\pi^1, \pi^2, \ldots, \pi^n$

Solve the first-stage problem

Do you recommend this option?

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Two-stage Deterministic Problem

*DDP functioning procedure*

How to generate possible values for $\pi^*$, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

**Option 2 (systematic iterative approach):**
Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for $\pi^*$, i.e., $\pi^1, \pi^2, ..., \pi^n$?

Option 2 (systematic iterative approach):

Iteration $i = 1$

Generate a trial value for $x_1$

Solve the second-stage problem for given $x_1$ and obtain $x_2$ and $\pi_i$:

$$\min_{x_2} c_2 x_2$$
subject to
$$A_2 x_2 \geq b_2 - E_1 x_1 : \pi_i$$

Solve the first-stage problem for given $\pi_i$ and update $x_1$:

$$\min_{\alpha, x_1} c_1 x_1 + \alpha$$
subject to
$$A_1 x_1 \geq b_1$$
$$\alpha \geq \pi_k (b_2 - E_1 x_1) \quad \forall k = 1, ..., i$$

Check $\alpha = c_2 x_2$

Yes
Optimal results

No

Updated $x_1$

$i = i + 1$

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Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for $\pi^*$, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 2 (systematic iterative approach):

Master problem

Iteration $i = 1$

Generate a trial value for $x_1$

Subproblem

Solve the second-stage problem for given $x_1$ and obtain $x_2$ and $\pi_i$:

$$\begin{align*}
\min_{x_2} \quad & c_2x_2 \\
\text{subject to} \quad & A_2x_2 \geq b_2 - E_1x_1 : \pi_i
\end{align*}$$

Solve the first-stage problem for given $\pi_i$ and update $x_1$:

$$\begin{align*}
\min_{\alpha, x_1} \quad & c_1x_1 + \alpha \\
\text{subject to} \quad & A_1x_1 \geq b_1 \\
& \alpha \geq \pi_k(b_2 - E_1x_1) \quad \forall k = 1, \ldots, i
\end{align*}$$

Check $\alpha = c_2x_2$

$\alpha = c_2x_2$

Optimal results

Updated $x_1$

$i = i + 1$

No

Yes

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Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for $\pi^*, \text{ i.e., } \pi^1, \pi^2, \ldots, \pi^n$?

Option 2 (systematic iterative approach):

This approach is indeed **Benders’ decomposition**!
Two-stage Deterministic Problem

DDP functioning procedure

Important note:

We can guarantee obtaining the global optimal solution by Benders’ decomposition, if the objective function of the original (non-decomposed) problem is convex with respect to the complicating variable!
Two-stage Deterministic Problem

Simple Example

\[
\begin{align*}
\text{minimize} & \quad z = -y - x/4 \\
\text{subject to} & \quad y - x \leq 5 \\
& \quad y - \frac{1}{2} x \leq \frac{15}{2} \\
& \quad y + \frac{1}{2} x \leq \frac{35}{2} \\
& \quad -y + x \leq 10 \\
& \quad 0 \leq x \leq 16 \\
& \quad y \geq 0.
\end{align*}
\]
Two-stage Deterministic Problem

Simple Example

Let’s consider $x$ as the complicating variable!

\[
\begin{align*}
\text{minimize} & \quad z = -y - x/4 \\
\text{subject to} & \quad y - x \leq 5 \\
& \quad y - \frac{1}{2}x \leq \frac{15}{2} \\
& \quad y + \frac{1}{2}x \leq \frac{35}{2} \\
& \quad -y + x \leq 10 \\
& \quad 0 \leq x \leq 16 \\
& \quad y \geq 0.
\end{align*}
\]
Two-stage Deterministic Problem

Simple Example

**Subproblem:**

\[
\begin{align*}
\text{minimize} & \quad -y^{(i)} \\
\text{subject to} & \\
y^{(i)} & \leq 5 + x^{\text{fixed}(i)} : \pi^{(i)} \\
y^{(i)} & \leq \frac{15}{2} + \frac{x^{\text{fixed}(i)}}{2} : \mu^{(i)} \\
y^{(i)} & \leq \frac{35}{2} - \frac{x^{\text{fixed}(i)}}{2} : \sigma^{(i)} \\
y^{(i)} & \leq 10 - x^{\text{fixed}(i)} : \gamma^{(i)} \\
y^{(i)} & \geq 0
\end{align*}
\]

**Master problem:**

\[
\begin{align*}
\text{minimize} & \quad -\frac{x^{(i)}}{4} + \alpha^{(i)} \\
\text{subject to} & \\
0 & \leq x^{(i)} \leq 16 \\
\alpha^{(i)} & \geq \alpha^{\text{down}} \\
\alpha^{(i)} & \geq \pi^{(k)}[5 + x^{(i)}] + \mu^{(k)}\left[\frac{15}{2} + \frac{x^{(i)}}{2}\right] + \sigma^{(k)}\left[\frac{35}{2} - \frac{x^{(i)}}{2}\right] \\
& + \gamma^{(k)}[10 - x^{(i)}] \quad \forall k = 1, \ldots, i - 1
\end{align*}
\]

\(i:\) current Benders’ iteration
\(k:\) set of previous iterations

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Two-stage Deterministic Problem

Simple Example

Subproblem:

\[
\begin{align*}
\text{minimize} & \quad -y^{(i)} \\
\text{subject to} & \\
y^{(i)} & \leq 5 + x^{\text{fixed}}(i) : \pi^{(i)} \\
y^{(i)} & \leq \frac{15}{2} + \frac{x^{\text{fixed}}(i)}{2} : \mu^{(i)} \\
y^{(i)} & \leq \frac{35}{2} - \frac{x^{\text{fixed}}(i)}{2} : \sigma^{(i)} \\
y^{(i)} & \leq 10 - x^{\text{fixed}}(i) : \gamma^{(i)} \\
y^{(i)} & \geq 0
\end{align*}
\]

i : current Benders' iteration
k : set of previous iterations

Note: In subproblem, symbols following colon are dual variables (sensitivities).

Master problem:

\[
\begin{align*}
\text{minimize} & \quad -\frac{x^{(i)}}{4} + \alpha^{(i)} \\
\text{subject to} & \\
0 & \leq x^{(i)} \leq 16 \\
\alpha^{(i)} & \geq \alpha^{\text{down}} \\
\alpha^{(i)} & \geq \pi^{(k)} \left[ 5 + x^{(i)} \right] + \mu^{(k)} \left[ \frac{15}{2} + \frac{x^{(i)}}{2} \right] + \sigma^{(k)} \left[ \frac{35}{2} - \frac{x^{(i)}}{2} \right] \\
& \quad + \gamma^{(k)} \left[ 10 - x^{(i)} \right] \quad \forall k = 1, \ldots, i - 1
\end{align*}
\]

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Two-stage Deterministic Problem

**Simple Example**

**Subproblem:**

\[
\text{minimize } -y^{(i)}
\]
subject to
\[
y^{(i)} \leq 5 + x^{\text{fixed (i)}} : \pi^{(i)}
\]
\[
y^{(i)} \leq \frac{15}{2} + \frac{x^{\text{fixed (i)}}}{2} : \mu^{(i)}
\]
\[
y^{(i)} \leq \frac{35}{2} - \frac{x^{\text{fixed (i)}}}{2} : \sigma^{(i)}
\]
\[
y^{(i)} \leq 10 - x^{\text{fixed (i)}} : \gamma^{(i)}
\]
\[
y^{(i)} \geq 0
\]

**Master problem:**

\[
\text{minimize } -\frac{x^{(i)}}{4} + \alpha^{(i)}
\]
subject to
\[
0 \leq x^{(i)} \leq 16
\]
\[
\alpha^{(i)} \geq \alpha^{\text{down}}
\]
\[
\alpha^{(i)} \geq \pi^{(k)}[5 + x^{(i)}] + \mu^{(k)}\left[\frac{15}{2} + \frac{x^{(i)}}{2}\right] + \sigma^{(k)}\left[\frac{35}{2} - \frac{x^{(i)}}{2}\right] + \gamma^{(k)}[10 - x^{(i)}] \quad \forall k = 1, \ldots, i - 1
\]

\[i: \text{ current Benders’ iteration}
\]
\[k: \text{ set of previous iterations}
\]

Note: In subproblem, symbols following colon are dual variables (sensitivities).

Note: The last constraint of master problem generate “cuts”.

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Two-stage Deterministic Problem

Simple Example

Algorithm:

• **Step 0: Initialization**
  Set $i = 1$, $x^{\text{fixed (1)}} = x^{\text{initial}}$, and lower bound (LB) = $-\infty$

• **Step 1: Solve subproblem(s):** obtain the values of all dual variables (sensitivities), and the value of objective function, which is upper bound (UB)

• **Step 2: Convergence check**
  If $|\text{UB} - \text{LB}| \leq \epsilon$, then the optimal solution with a level of accuracy $\epsilon$ is obtained, otherwise $i \leftarrow i + 1$

• **Step 3: Solve master problem:** obtain the updated $x^{(i)}$ and the value of $\alpha^{(i)}$ as the updated LB, and go Step 1 with the updated fixed $x$

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A more compact form of Benders’ decomposition:

**Subproblem:**

\[
\begin{align*}
\text{minimize} & \quad -y^{(i)} \\
\text{subject to} & \\
y^{(i)} - x^{(i)} & \leq 5 \\
y^{(i)} - \frac{x^{(i)}}{2} & \leq \frac{15}{2} \\
y^{(i)} + \frac{x^{(i)}}{2} & \leq \frac{35}{2} \\
y^{(i)} + x^{(i)} & \leq 10 \\
y^{(i)} & \geq 0 \\
x^{(i)} & = x^{\text{fixed}}(i) : \rho^{(i)}
\end{align*}
\]

**Master problem:**

\[
\begin{align*}
\text{minimize} & \quad -\frac{x^{(i)}}{4} + \alpha^{(i)} \\
\text{subject to} & \\
\alpha^{(i)} & \geq \alpha^{\text{down}} \\
\alpha^{(i)} & \geq -y^{(k)} + \rho^{(k)}[x^{(i)} - x^{(k)}] \quad \forall k = 1, \ldots, i - 1
\end{align*}
\]
Two-stage **Stochastic** Problem

*Two-stage (day-ahead and real-time) stochastic OPF problem*
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

Day-ahead (DA) market → Real-time (RT) market

DA decisions

RT decisions for scenario 1
RT decisions for scenario 2
•
•
RT decisions for scenario n

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Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

Day-ahead (DA) decisions:

Real-time (RT) decisions:
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

Day-ahead (DA) decisions:
• Power schedule [MW] of each generator (\( \forall g: 1, \ldots, G \)), which is \( P_g \).

Note: this variable is scenario-independent, in the sense that it should be adapted to all foreseen scenarios (here-and-now decisions).

Real-time (RT) decisions:
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

Day-ahead (DA) decisions:
• Power schedule [MW] of each generator ($\forall g: 1, \ldots, G$), which is $P_g$.

Note: this variable is scenario-independent, in the sense that it should be adapted to all foreseen scenarios (here-and-now decisions).

Real-time (RT) decisions:
(For given DA decisions)
• Reserve deployment [MW] of each generator $g$ under each foreseen scenario ($\forall s: 1, \ldots, S$), which is $r_{g,s}$.
• Load shedding, wind curtailment, etc (all indexed by $s$).

Note: this variable is scenario-dependent (wait-and-see decisions).
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

- Is there any complicating variable in this two-stage stochastic problem?

```
DA decisions
  └── RT decisions for scenario 1
  │     │
  │     └── RT decisions for scenario 2
  │            └──
  │                         └── RT decisions for scenario n
```
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

- Is there any complicating variable in this two-stage stochastic problem?

Fix DA decisions to given values!
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

- Is there any complicating variable in this two-stage stochastic problem?

DA decisions

Fix DA decisions to given values!

RT decisions for scenario 1
RT decisions for scenario 2
RT decisions for scenario \( n \)
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

• Is there any complicating variable in this two-stage stochastic problem?

DA decisions

Fix DA decisions to given values!

Then, how many subproblems will you have?

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Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

- Is there any complicating variable in this two-stage stochastic problem?

DA decisions

\[ \rightarrow \text{RT decisions for scenario 1} \]
\[ \rightarrow \text{RT decisions for scenario 2} \]
\[ \rightarrow \text{RT decisions for scenario } n \]

Fix DA decisions to given values! Then, how many subproblems will you have? **One per scenario!**
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

Master problem

Day-ahead (DA) decision-making problem (DA-OPF)

Real-time (RT) decision-making problems (RT-OPF), one per scenario

Subproblems (one per scenario)

Fixed DA variables

Sensitivities (dual variables)
The application of Benders’ decomposition to two-stage stochastic problems is also referred in the literature as **L-shaped decomposition**!
Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?
What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

- **Step 1 in each iteration:** Solve a single two-stage stochastic problem
- **Master problem:** Stage 1
- **Subproblems:** Stage 2
Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

- **Step 2 in each iteration:** Solve 3 two-stage stochastic problems (separately)
- **Master problems:** Stage 2
- **Subproblems:** Stage 3
What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

- **Step 3 in each iteration**: Solve 9 two-stage stochastic problems (separately)
- **Master problems**: Stage 3
- **Subproblems**: Stage 4
Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

Each multi-stage stochastic problem is a collection of **nested** two-stage stochastic problems.
Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

Each multi-stage stochastic problem is a collection of nested two-stage stochastic problems.

Techniques to be used:
• Nested Benders’ decomposition
• Stochastic dual dynamic programming (SDDP)

Some more thoughts

What to do in a case in which subproblems include \texttt{binary (0/1)} variables?
Some more thoughts

What to do in a case in which subproblems include **binary (0/1)** variables?

**The trouble?**

Sensitivities (dual variables) cannot be obtained in a discrete feasible region! But, we need those sensitivities to generate cuts in the master problem!
Some more thoughts

What to do in a case in which subproblems include *binary (0/1)* variables?

**Solution:**

- Generate cuts in the master problem based on the values obtained for “primal” variables, and not “dual” variables of the subproblems!
Some more thoughts

What to do in a case in which subproblems include **binary (0/1)** variables?

**Solution technique:**

- Primal Benders’ decomposition (cutting-plane method)

  This technique has recently been used in power systems applications for two-stage robust optimization problems

References


Additional References


Thanks for your attention!

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