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Technical University of Denmark, Lyngby, Denmark

EES-UETP Course title

Decomposition techniques for optimization problems with complicating constraints

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Learning Objectives

DTU

After this session the participants are expected to be able to:

- Explain the functioning of Lagrangian relaxation (LR), augmented Lagrangian relaxation (ALR), and alternating direction method of multipliers (ADMM)

Decomposition Techniques

Applicable to optimization problems with complicating constraints

- Lagrangian relaxation (LR)

  In the literature, this technique has been also known as ***standard*** or ***conventional*** LR!

- Augmented Lagrangian relaxation (ALR)
  - ❑ Auxiliary problem principle (APP)
  - ❑ Alternating direction method of multipliers (ADMM)

- Dantzig-Wolfe decomposition (DWD)

- …

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Decomposition Techniques

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- Dantzig-Wolfe decomposition (DWD)
- …

Will not be covered in this course!

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Optimization problems with complicating constraints

Some examples related to power systems

- **Single-node optimal power flow (OPF) problem (investigated in the previous lecture)**
  - ✓ **Complicating constraints**: balance equalities and ramping limits of generators
  - ✓ If relaxed, original problem decomposes by agent (and hour)

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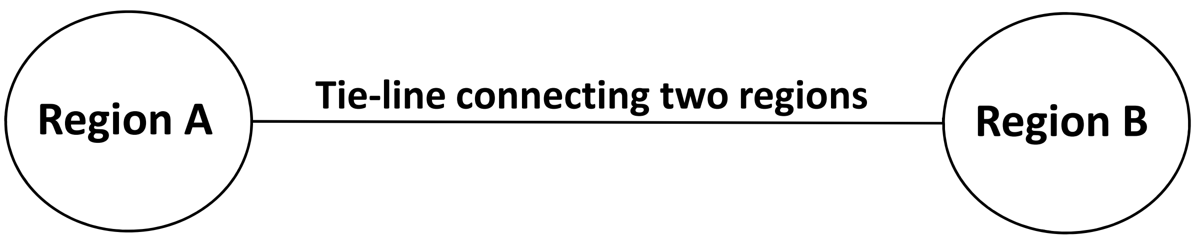
Optimization problems with complicating constraints

Some examples related to power systems

- **Single-node optimal power flow (OPF) problem (investigated in the previous lecture)**

  ✓ **Complicating constraints**: balance equalities and ramping limits of generators
  ✓ If relaxed, original problem decomposes by agent (and hour)

- **Multi-regional OPF (or unit commitment) problem, e.g., in case of pan-European electricity market**



  ✓ **Complicating constraints**: tie-line constraints (power flow, and tie-line capacity)
  ✓ If relaxed, original problem decomposes by region. This way, operator of each region only solves its own OPF problem (so-called ***distributed*** OPF problem)

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Lagrangian Relaxation (LR)

Background

- The theory of LR (and also ALR) was firstly developed for problems with continuous variables, and functions (objective function and constraints) with first derivatives continuous.

- However, the theory has been used in problems with binary variables (like unit commitment problems) with success.

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Lagrangian Relaxation (LR)

Background

- LR works efficiently if the number of complicating constraints is <u>relatively low</u>, and it is OK to have binary variables in the formulation.

- LR was extensively used in the 90's to solve unit commitment problems (complicating constrains are just balance constraints and ramping constraints).

Key point

In case of LR:

In addition to convexity, the objective function of the original problem (not decomposed problem) needs to be smooth (continuous first derivatives), e.g., quadratic. If this objective function is linear, the LR procedure does <u>not</u> converge!

- Alternative solution technique for problems with linear objective function is ALR.

Lagrangian Relaxation (LR)

Background

For unit commitment (and also OPF) problems:

- LR (for problems with quadratic objective function)
- ALR (for problems with either quadratic or linear objective function)

Both have been extensively and very successfully used in the literature, though unit commitment problem is fully non-convex (due to binary variables).

Lagrangian Relaxation (LR)

Mathematical procedure

Original (non-decomposed) problem:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i)$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$

$$h_i(x_i) \leq B_i \qquad \forall i$$

$$\sum_{i=1}^{I} c_i(x_i) = M$$

$$\sum_{i=1}^{I} d_i(x_i) \leq N$$

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Lagrangian Relaxation (LR)

Mathematical procedure

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Complicating constraints

$$\sum_{i=1}^{I} c_i(x_i) = M \qquad (\lambda)$$

$$\sum_{i=1}^{I} d_i(x_i) \leq N \qquad (\mu)$$

Dual variables (Lagrangian multipliers)

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In the optimal point, the original problem is equivalent to:

$$\underset{x_i,\lambda,\mu}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i) + \lambda \left[M - \sum_{i=1}^{I} c_i(x_i) \right] + \mu \left[N - \sum_{i=1}^{I} d_i(x_i) \right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$
$$h_i(x_i) \leq B_i \qquad \forall i$$

Lagrangian Relaxation (LR)

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Is the original problem decomposable now?

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Subject to

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Is the original problem decomposable now? Not yet!

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Mathematical procedure

In the optimal point, the original problem is equivalent to:

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Subject to

$$g_i(x_i) = A_i \qquad \forall i$$
$$h_i(x_i) \leq B_i \qquad \forall i$$

Is the original problem decomposable now? Not yet!

- Let's **relax** the equivalent problem above by **fixing dual variables (λ and μ)** to given values, i.e., $\bar{\lambda}$ **and** $\bar{\mu}$.

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In the optimal point, the original problem is equivalent to:

$$\underset{x_i}{\text{Minimize}} \quad \sum_{i=1}^{I} f_i(x_i) + \bar{\lambda}\left[M - \sum_{i=1}^{I} c_i(x_i)\right] + \bar{\mu}\left[N - \sum_{i=1}^{I} d_i(x_i)\right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$
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Is the original problem decomposable now?

In the optimal point, the original problem is equivalent to:

$$\underset{x_i}{\text{Minimize}} \;\; \sum_{i=1}^{I} f_i(x_i) + \bar{\lambda}\left[M - \sum_{i=1}^{I} c_i(x_i)\right] + \bar{\mu}\left[N - \sum_{i=1}^{I} d_i(x_i)\right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$
$$h_i(x_i) \leq B_i \qquad \forall i$$

Is the original problem decomposable now? Yes, one per i:

$$\left\{ \underset{x_i}{\text{Minimize}} \;\; f_i(x_i) + \bar{\lambda}\, c_i(x_i) + \bar{\mu}\, d_i(x_i) \right.$$

Subject to

$$\left.\begin{array}{l} g_i(x_i) = A_i \\ h_i(x_i) \leq B_i \end{array}\right\} \forall i$$

Lagrangian Relaxation (LR)

Mathematical procedure

LR is an iterative approach with a systematic way to update the values of fixed dual variables ($\bar{\lambda}$ and $\bar{\mu}$) in each iteration.

Available techniques in the literature to update $\bar{\lambda}$ and $\bar{\mu}$:

1. Subgradient method
2. Cutting plane method
3. Bundle method
4. Trust region method
5. …

Lagrangian Relaxation (LR)

Mathematical procedure

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Will not be covered in this course!

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Lagrangian Relaxation (LR)

Numerical example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

$$\text{Subject to} \quad -x - y \leq -4 \qquad (\mu)$$

Note: Objective function includes quadratic terms, so LR works!

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Lagrangian Relaxation (LR)

Numerical example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

Subject to $\quad -x - y \leq -4 \qquad (\mu)$

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Subproblem 1:

$$\underset{x\geq 0}{\text{Minimize}} \; x^2 - \bar{\mu}x$$

Subproblem 2:

$$\underset{y\geq 0}{\text{Minimize}} \; y^2 - \bar{\mu}y$$

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Lagrangian Relaxation (LR)

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$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

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Subproblem 1:

$$\underset{x\geq 0}{\text{Minimize}} \; x^2 - \bar{\mu}x$$

Subproblem 2:

$$\underset{y\geq 0}{\text{Minimize}} \; y^2 - \bar{\mu}y$$

Updating fixed dual variable ($\bar{\mu}$) using subgradient method:

- Solve subproblems 1 and 2 in iteration v, and obtain the values $x^{(v)}$ and $y^{(v)}$
- $\bar{\mu}^{(v+1)} \leftarrow \bar{\mu}^{(v)} + \frac{1}{a+bv} \frac{-x^{(v)}-y^{(v)}+4}{\left|-x^{(v)}-y^{(v)}+4\right|}$
- a and b are positive constants, e.g., $a = 1$ and $b = 0.1$.

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Lagrangian Relaxation (LR)

Numerical example

Algorithm:

- **Step 0: Initialization**

  Set $\upsilon = 1$ and $\bar{\mu}^{(1)} = \bar{\mu}^{\text{initial}}$

- **Step 1: Solve subproblems 1 and 2, and obtain $\boldsymbol{x}^{(\boldsymbol{\upsilon})}$ and $\boldsymbol{y}^{(\boldsymbol{\upsilon})}$**

- **Step 2: Update fixed dual variable, i.e., $\overline{\boldsymbol{\mu}}^{(\boldsymbol{\upsilon}+\mathbf{1})}$**

- **Step 3: Convergence check**

  If $\frac{\left\|\overline{\mu}^{(\upsilon+1)}-\overline{\mu}^{(\upsilon-1)}\right\|}{\left\|\overline{\mu}^{(\upsilon)}\right\|} \leq \epsilon$, then the optimal solution with a level of accuracy ϵ is obtained, otherwise $\upsilon \leftarrow \upsilon + 1$ and go Step 1

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Lagrangian Relaxation (LR)

Numerical example

Exercise:

The printed version of GAMS code of the LR example is available on your table. Please check it in the next 15 minutes, then explain it to your neighbors around the table.

This code has been prepared by Lejla Halilbasic and Christos Ordoudis.

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Augmented Lagrangian Relaxation (ALR)

Functioning procedure

Recall:

ALR works for problems with either quadratic objective function (like LR) or linear one (unlike LR)

Main difference of ALR with respect to LR:

An additional penalty term within the subproblems

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

Subject to $-x - y = -4 \;\; (\lambda)$

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}}\ x^2 + y^2$$

Subject to $-x - y = -4 \quad (\lambda)$

Additional penalty term with respect to LR, γ is a positive constant

Equivalent to:

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}}\ x^2 + y^2 + \lambda(-x - y + 4) + \frac{\gamma}{2}\|-x - y + 4\|^2$$

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

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Question:

- **Similar to LR, assume dual variable λ is fixed to given value $\bar{\lambda}$. Is the problem above decomposable for given $\bar{\lambda}$?**

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

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Question:

- **Similar to LR, assume dual variable λ is fixed to given value $\bar{\lambda}$. Is the problem above decomposable for given $\bar{\lambda}$? No, due to product of x and y in the penalty term!**

Augmented Lagrangian Relaxation (ALR)

Numerical Example

Available alternatives to solve ALR:

- Auxiliary problem principle (APP)
- Alternating direction method of multipliers (ADMM)

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

Available alternatives to solve ALR:

- Auxiliary problem principle (APP): *will not be covered in this course*
- **Alternating direction method of multipliers (ADMM)**

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

Available alternatives to solve ALR:

- Auxiliary problem principle (APP): *will not be covered in this course*
- **<u>Alternating direction method of multipliers (ADMM)</u>**

Note:

ADMM directly fixes variables to their values in the previous iteration, and decomposes the ALR to subproblems.

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

Available alternatives to solve ALR:

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- **Alternating direction method of multipliers (ADMM)**

Note:

ADMM directly fixes variables to their values in the previous iteration, and decomposes the ALR to subproblems.

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2 + \lambda(-x - y + 4) + \frac{\gamma}{2}\|-x - y + 4\|^2$$

The problem above in iteration υ can be decomposed to two subproblems:

$$\begin{cases} \underset{x^{(\upsilon)}\geq 0}{\text{Minimize}} \; x^{2(\upsilon)} + \lambda^{(\upsilon-1)}(-x^{(\upsilon)}+2) + \frac{\gamma}{2}\left\|-x^{(\upsilon)} - y^{(\upsilon-1)} + 4\right\|^2 \\ \underset{y^{(\upsilon)}\geq 0}{\text{Minimize}} \; y^{2(\upsilon)} + \lambda^{(\upsilon-1)}(-y^{(\upsilon)}+2) + \frac{\gamma}{2}\left\|-y^{(\upsilon)} - x^{(\upsilon-1)} + 4\right\|^2 \end{cases}$$

where $\lambda^{(\upsilon)} \leftarrow \lambda^{(\upsilon-1)} + \gamma\left(-x^{(\upsilon)} - y^{(\upsilon)} + 4\right)$

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Exercise:

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References

- A. J. Conejo, E. Castillo, R. Minguez, and R. Garcia-Bertrand, *Decomposition Techniques in Mathematical Programming: Engineering and Science Applications*. Berlin, Germany: Springer, 2006.

- S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, “Distributed optimization and statistical learning via the alternating direction method of multipliers,” Foundations and Trends in Machine Learning, vol. 3, no. 1, pp. 1-122, Jan. 2011.

Additional References

- Y. Zhang and G. B. Giannakis, “Distributed stochastic market clearing with high penetration wind power,” IEEE Trans. Power Syst., to be published, 2016.

- A. Ahmadi-Khatir, A. J. Conejo, and R. Cherkaoui, “Multi-area unit scheduling and reserve allocation under wind power uncertainty,” IEEE Trans. Power Syst., vol. 29, no. 4, pp. 1701-1710, Jul. 2014.

- N. J. Redondo and A. J. Conejo, “Short-term hydro-thermal coordination by Lagrangian relaxation: Solution of the dual problem,” IEEE Trans. Power Syst., vol. 14, no. 1, pp. 89-95, Feb. 1999.

- Q. Peng and S. H. Low, “Distributed optimal power flow algorithm for radial networks, I: Balanced single phase case,” IEEE Trans. Smart Grid, to be published, 2016.

Thanks for your attention!

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