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EES-UETP Course title

Optimization problems with Decomposable structure

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Learning Objectives

After this session the participants are expected to be able to:

- Explain the need for decomposition
- Identify whether each optimization problem is decomposable or not (if so, how?)

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Decomposition

Main idea

Original (non-decomposed) optimization problem with decomposable structure

↓ **Decomposition**

Decomposed optimization problem 1 | **Decomposed optimization problem 2** | • • • | **Decomposed optimization problem *n***

Each decomposed problem is easier-to-solve than the original (non-decomposed) problem!

Decomposition

Motivation

- Why do we need decomposition in power systems?

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Decomposition

Motivation

- Why do we need decomposition in power systems?
  - Operation problems (e.g., unit commitment)
  - Planning problems (e.g., expansion)

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Decomposition

Motivation

- Why do we need decomposition in power systems?
  - Operation problems (e.g., unit commitment)
    - Need to be computationally tractable
    - Need to be solved in a specific time period
  - Planning problems (e.g., expansion)
    - Need to be computationally tractable

Decomposition

Decomposable structure

Optimization problems with decomposable structure:

- Problems with complicating variable(s):

- Problems with complicating constraint(s):

Decomposition

Decomposable structure

Optimization problems with decomposable structure:

- Problems with complicating variable(s):

  The original problem is decomposed if the complicating variables are fixed to given values!

- Problems with complicating constraint(s):

  The original problem is decomposed if the complicating constraints are relaxed (removed)!

Decomposition

Decomposable structure

Optimization problems with decomposable structure:

- Problems with complicating variable(s):

  The original problem is decomposed if the complicating variables are fixed to given values!

- Problems with complicating constraint(s):

  The original problem is decomposed if the complicating constraints are relaxed (removed)!

In the literature, “complicating” variables/constraints are also called as “coupling” variables/constraint!

Decomposition

Features of decomposition techniques

- Iterative solution techniques

- Original optimization problem decomposes to:
  - A single master problem (not always!)
  - A set of subproblems

Decomposable Structures

Optimization problems with complicating constraint(s)

Example: A linear programming (LP) as the original problem

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Decomposable Structures

Optimization problems with complicating constraint(s)

Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,x_3,\\ y_1,y_2,\\ z_1,z_2,z_3}}{\text{Minimize}} \quad A_1x_1 + A_2x_2 + A_3x_3 + B_1y_1 + B_2y_2 + C_1z_1 + C_2z_2 + C_3z_3$$

Subject to

$$E_{11}x_1 + E_{12}x_2 + E_{13}x_3 \geq F_1$$
$$E_{21}x_1 + E_{22}x_2 + E_{23}x_3 \geq F_2$$

$$E_{31}y_1 + E_{32}y_2 \geq F_3$$

$$E_{41}z_1 + E_{42}z_2 + E_{43}z_3 \geq F_4$$
$$E_{51}z_1 + E_{52}z_2 + E_{53}z_3 \geq F_5$$

$$E_{61}x_1 + E_{62}x_2 + E_{63}x_3 + E_{64}y_1 + E_{65}y_2 + E_{66}z_1 + E_{67}z_2 + E_{68}z_3 \geq F_6$$

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Decomposable Structures

Optimization problems with complicating constraint(s)

Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,x_3,\\ y_1,y_2,\\ z_1,z_2,z_3}}{\text{Minimize}} \quad A_1x_1 + A_2x_2 + A_3x_3 + B_1y_1 + B_2y_2 + C_1z_1 + C_2z_2 + C_3z_3$$

Subject to

$$E_{11}x_1 + E_{12}x_2 + E_{13}x_3 \geq F_1$$
$$E_{21}x_1 + E_{22}x_2 + E_{23}x_3 \geq F_2$$

$$E_{31}y_1 + E_{32}y_2 \geq F_3$$

$$E_{41}z_1 + E_{42}z_2 + E_{43}z_3 \geq F_4$$
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$$E_{61}x_1 + E_{62}x_2 + E_{63}x_3 + E_{64}y_1 + E_{65}y_2 + E_{66}z_1 + E_{67}z_2 + E_{68}z_3 \geq F_6$$

Decomposable Structures

Optimization problems with complicating constraint(s)

Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,x_3,\\ y_1,y_2,\\ z_1,z_2,z_3}}{\text{Minimize}} \quad A_1x_1 + A_2x_2 + A_3x_3 + B_1y_1 + B_2y_2 + C_1z_1 + C_2z_2 + C_3z_3$$

Subject to

Constraints including only variables x

$$E_{11}x_1 + E_{12}x_2 + E_{13}x_3 \geq F_1$$
$$E_{21}x_1 + E_{22}x_2 + E_{23}x_3 \geq F_2$$

Constraint including only variables y

$$E_{31}y_1 + E_{32}y_2 \geq F_3$$

Constraints including only variables z

$$E_{41}z_1 + E_{42}z_2 + E_{43}z_3 \geq F_4$$
$$E_{51}z_1 + E_{52}z_2 + E_{53}z_3 \geq F_5$$

$$E_{61}x_1 + E_{62}x_2 + E_{63}x_3 + E_{64}y_1 + E_{65}y_2 + E_{66}z_1 + E_{67}z_2 + E_{68}z_3 \geq F_6$$

Decomposable Structures

Optimization problems with complicating constraint(s)

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Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,x_3,\\ y_1,y_2,\\ z_1,z_2,z_3}}{\text{Minimize}} \quad A_1x_1 + A_2x_2 + A_3x_3 + B_1y_1 + B_2y_2 + C_1z_1 + C_2z_2 + C_3z_3$$

Subject to

Constraints including only variables x

$$E_{11}x_1 + E_{12}x_2 + E_{13}x_3 \geq F_1$$
$$E_{21}x_1 + E_{22}x_2 + E_{23}x_3 \geq F_2$$

Constraint including only variables y

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$$E_{41}z_1 + E_{42}z_2 + E_{43}z_3 \geq F_4$$
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$$E_{61}x_1 + E_{62}x_2 + E_{63}x_3 + E_{64}y_1 + E_{65}y_2 + E_{66}z_1 + E_{67}z_2 + E_{68}z_3 \geq F_6$$

Decomposable Structures

Optimization problems with complicating constraint(s)

Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,x_3,\\ y_1,y_2,\\ z_1,z_2,z_3}}{\text{Minimize}} \quad A_1x_1 + A_2x_2 + A_3x_3 + B_1y_1 + B_2y_2 + C_1z_1 + C_2z_2 + C_3z_3$$

Subject to

Constraints including only variables x

$$E_{11}x_1 + E_{12}x_2 + E_{13}x_3 \geq F_1$$
$$E_{21}x_1 + E_{22}x_2 + E_{23}x_3 \geq F_2$$

Constraint including only variables y

$$E_{31}y_1 + E_{32}y_2 \geq F_3$$

Constraints including only variables z

$$E_{41}z_1 + E_{42}z_2 + E_{43}z_3 \geq F_4$$
$$E_{51}z_1 + E_{52}z_2 + E_{53}z_3 \geq F_5$$

$$E_{61}x_1 + E_{62}x_2 + E_{63}x_3 + E_{64}y_1 + E_{65}y_2 + E_{66}z_1 + E_{67}z_2 + E_{68}z_3 \geq F_6$$

This is complicating constraint: If relaxed (removed), then the original problem decomposes to three smaller optimization problems (subproblems)!

Example: A linear programming (LP) as the original problem

Subproblem 1:

$$\underset{x_1, x_2, x_3}{\text{Minimize}} \quad A_1 x_1 + A_2 x_2 + A_3 x_3$$

Subject to

Constraints including only variables x

$$E_{11} x_1 + E_{12} x_2 + E_{13} x_3 \geq F_1$$
$$E_{21} x_1 + E_{22} x_2 + E_{23} x_3 \geq F_2$$

Example: A linear programming (LP) as the original problem

Subproblem 2:

$$\underset{y_1, y_2}{\text{Minimize}} \quad B_1 y_1 + B_2 y_2$$

Subject to

Constraint including only variables y

$$E_{31} y_1 + E_{32} y_2 \geq F_3$$

Example: A linear programming (LP) as the original problem

Subproblem 3:

$$\underset{z_1, z_2, z_3}{\text{Minimize}} \quad C_1 z_1 + C_2 z_2 + C_3 z_3$$

Subject to

Constraints including only variables z

$$E_{41} z_1 + E_{42} z_2 + E_{43} z_3 \geq F_4$$
$$E_{51} z_1 + E_{52} z_2 + E_{53} z_3 \geq F_5$$

Decomposable Structures

Optimization problems with complicating constraint(s)

Example: A linear programming (LP) as the original problem

Decomposable matrix with complicating constraint(s)

$$\begin{bmatrix} A^T & B^T & C^T \end{bmatrix}$$

Subject to

$$\begin{bmatrix} E^{[1]} & & \\ & E^{[2]} & \\ & & E^{[3]} \\ E^{[5]} & E^{[6]} & E^{[7]} \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} F^{[1]} \\ F^{[2]} \\ F^{[3]} \\ F^{[4]} \end{bmatrix}$$

Complicating constraint

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Example: A linear programming (LP) as the original problem

Decomposable Structures

Optimization problems with complicating variable(s)

Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,\\ y_1,y_2,\\ z_1,z_2,\\ \beta}}{\text{Minimize}} \quad A_1x_1 + A_2x_2 + B_1y_1 + B_2y_2 + C_1z_1 + C_2z_2 + D_1\,\beta$$

Subject to

$$E_{11}x_1 + E_{12}\,x_2 + E_{13}\,\beta \geq F_1$$
$$E_{21}x_1 + E_{22}\,x_2 + E_{23}\beta \geq F_2$$

$$E_{31}y_1 + E_{32}\,y_2 + E_{33}\,\beta \geq F_3$$

$$E_{41}z_1 + E_{42}\,z_2 + E_{43}\,\beta \geq F_4$$
$$E_{51}z_1 + E_{52}\,z_2 + E_{53}\,\beta \geq F_5$$

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Decomposable Structures

Optimization problems with complicating variable(s)

Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,\\ y_1,y_2,\\ z_1,z_2,\\ \beta}}{\text{Minimize}} \quad A_1 x_1 + A_2 x_2 + B_1 y_1 + B_2 y_2 + C_1 z_1 + C_2 z_2 + D_1 \beta$$

Subject to

$$E_{11} x_1 + E_{12} x_2 + E_{13} \beta \geq F_1$$
$$E_{21} x_1 + E_{22} x_2 + E_{23} \beta \geq F_2$$

$$E_{31} y_1 + E_{32} y_2 + E_{33} \beta \geq F_3$$

$$E_{41} z_1 + E_{42} z_2 + E_{43} \beta \geq F_4$$
$$E_{51} z_1 + E_{52} z_2 + E_{53} \beta \geq F_5$$

Decomposable Structures

Optimization problems with complicating variable(s)

Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,\\ y_1,y_2,\\ z_1,z_2,\\ \beta}}{\text{Minimize}} \quad A_1 x_1 + A_2 x_2 + B_1 y_1 + B_2 y_2 + C_1 z_1 + C_2 z_2 + D_1 \beta$$

Subject to

$$E_{11} x_1 + E_{12} x_2 + E_{13} \beta \geq F_1$$
$$E_{21} x_1 + E_{22} x_2 + E_{23} \beta \geq F_2$$

$$E_{31} y_1 + E_{32} y_2 + E_{33} \beta \geq F_3$$

$$E_{41} z_1 + E_{42} z_2 + E_{43} \beta \geq F_4$$
$$E_{51} z_1 + E_{52} z_2 + E_{53} \beta \geq F_5$$

β is a complicating variable, i.e., if it is fixed to a given value (β^{fixed}), then the original problem decomposes to 3 smaller problems (subproblems)!

Decomposable Structures

Optimization problems with complicating variable(s)

Example: A linear programming (LP) as the original problem

$$\underset{\substack{x_1,x_2,\\ y_1,y_2,\\ z_1,z_2,\\ \beta}}{\text{Minimize}} \quad A_1 x_1 + A_2 x_2 + B_1 y_1 + B_2 y_2 + C_1 z_1 + C_2 z_2 + \underbrace{D_1\, \beta^{\text{fixed}}}$$

Fixed term, it can be removed from objective function

Subject to

Constraints including only variables x

$$E_{11} x_1 + E_{12} x_2 \geq F_1 - E_{13}\, \beta^{\text{fixed}}$$
$$E_{21} x_1 + E_{22} x_2 \geq F_2 - E_{23}\, \beta^{\text{fixed}}$$

Constraint including only variables y

$$E_{31} y_1 + E_{32} y_2 \geq F_3 - E_{33}\, \beta^{\text{fixed}}$$

Constraints including only variables z

$$E_{41} z_1 + E_{42} z_2 \geq F_4 - E_{43}\, \beta^{\text{fixed}}$$
$$E_{51} z_1 + E_{52} z_2 \geq F_5 - E_{53}\, \beta^{\text{fixed}}$$

β is a complicating variable, i.e., if it is fixed to a given value (β^{fixed}), then the original problem decomposes to 3 smaller problems (subproblems)!

Decomposable Structures

Optimization problems with complicating variable(s)

Example: A linear programming (LP) as the original problem

Subproblem 1:

$$\underset{x_1, x_2}{\text{Minimize}} \quad A_1 x_1 + A_2 x_2$$

Subject to

Constraints including only variables x

$$E_{11} x_1 + E_{12} x_2 \geq F_1 - E_{13}\, \beta^{\text{fixed}}$$
$$E_{21} x_1 + E_{22} x_2 \geq F_2 - E_{23}\, \beta^{\text{fixed}}$$

Example: A linear programming (LP) as the original problem

Subproblem 2:

$$\underset{y_1, y_2}{\text{Minimize}}\ B_1 y_1 + B_2 y_2$$

Subject to

Constraint including only variables y

$$E_{31} y_1 + E_{32} y_2 \geq F_3 - E_{33}\ \beta^{\text{fixed}}$$

Decomposable Structures

Optimization problems with complicating variable(s)

Example: A linear programming (LP) as the original problem

Subproblem 3:

$$\underset{z_1, z_2}{\text{Minimize}}\; C_1 z_1 + C_2 z_2$$

Subject to

Constraints including only variables z

$$E_{41} z_1 + E_{42} z_2 \geq F_4 - E_{43}\, \beta^{\text{fixed}}$$
$$E_{51} z_1 + E_{52} z_2 \geq F_5 - E_{53}\, \beta^{\text{fixed}}$$

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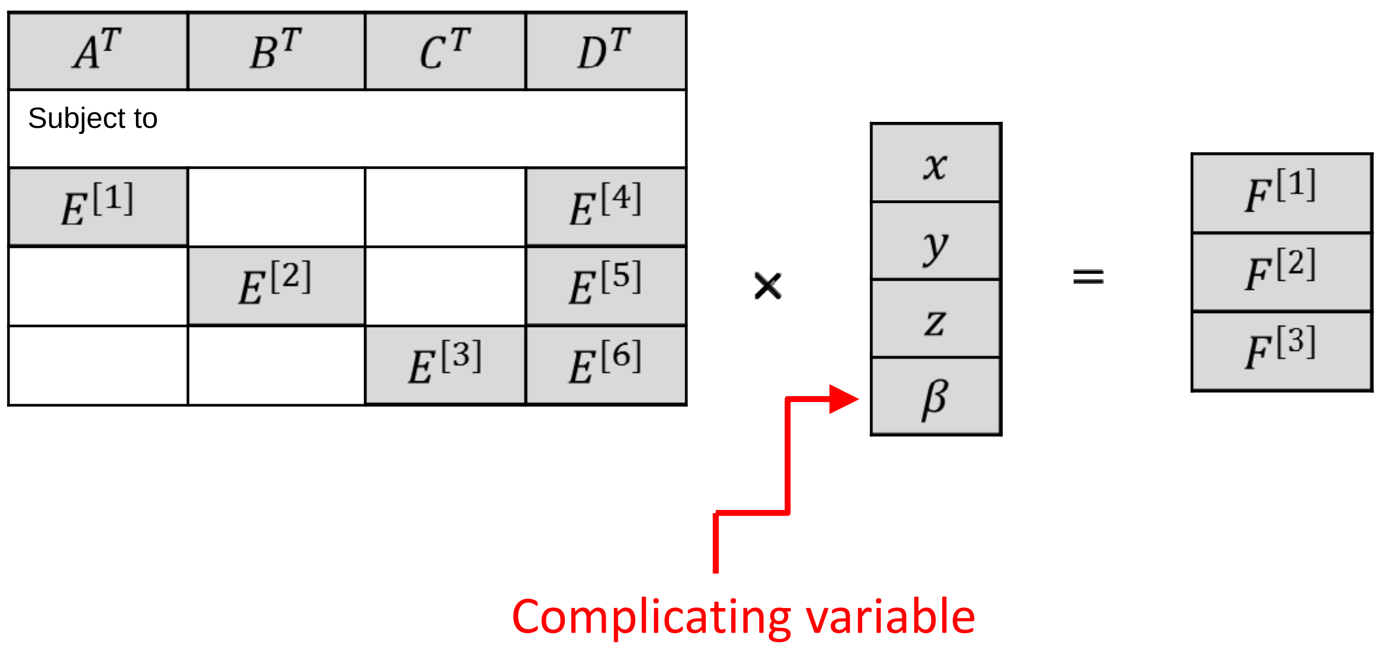
Decomposable Structures

Optimization problems with complicating variable(s)

Example: A linear programming (LP) as the original problem

Decomposable matrix with complicating variable(s)



Exercise:

Seven examples are available in the papers on your table. Please check them in the next 10 minutes, and identify whether they are decomposable problems or not (if so, how?). Then, check your results with your neighbors around the table.

Example 1

Identify whether the following problem is decomposable or not (if so, how)!

$$\underset{x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,w_1}{\text{Minimize}} \quad -4x_1 - y_1 - 6z_1$$

subject to

$$x_1 - x_2 = 1$$

$$x_1 + x_3 = 3$$

$$y_1 - y_2 = 1$$

$$y_1 + y_3 = 2$$

$$z_1 - z_2 = 1$$

$$z_1 + z_3 = 2$$

$$3x_1 + 2y_1 + 4z_1 + w_1 = 17$$

$$x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, w_1 \geq 0$$

Example 1

Identify whether the following problem is decomposable or not (if so, how)!

$$\underset{x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3,w_1}{\text{Minimize}} \quad -4x_1 - y_1 - 6z_1$$

subject to

$$x_1 - x_2 = 1$$

$$x_1 + x_3 = 3$$

$$y_1 - y_2 = 1$$

$$y_1 + y_3 = 2$$

$$z_1 - z_2 = 1$$

$$z_1 + z_3 = 2$$

$$\boxed{3x_1 + 2y_1 + 4z_1 + w_1 = 17}$$

$$x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, w_1 \geq 0$$

→ **Complicating constraint:** if relaxed, then the original problem decomposes to 3 subproblems

Example 2

Identify whether the following problem is decomposable or not (if so, how)!

$$\underset{x_1,x_2,x_3,x_4,x_5,x_6}{\text{Maximize}} \; 4x_1 + 3x_2 + 2x_3 + 3x_4 + 2x_5 + 3x_6$$

subject to

$$x_1 - 2x_2 + 2x_6 \leq 3$$

$$2x_1 + x_2 + x_6 \leq 3$$

$$-2x_1 + 3x_2 + x_6 \leq 7$$

$$x_3 + 3x_6 \leq 4$$

$$2x_3 - x_6 \leq 3$$

$$x_4 \leq 1$$

$$2x_4 - 4x_5 + 3x_6 \leq 5$$

$$3x_4 + x_5 - x_6 \leq 4$$

Example 2

Identify whether the following problem is decomposable or not (if so, how)!

$$\underset{x_1,x_2,x_3,x_4,x_5,x_6}{\text{Maximize}} \; 4x_1 + 3x_2 + 2x_3 + 3x_4 + 2x_5 + 3x_6$$

subject to

$$x_1 - 2x_2 + 2x_6 \leq 3$$

$$2x_1 + x_2 + x_6 \leq 3$$

$$-2x_1 + 3x_2 + x_6 \leq 7$$

$$x_3 + 3x_6 \leq 4$$

$$2x_3 - x_6 \leq 3$$

$$x_4 \leq 1$$

$$2x_4 - 4x_5 + 3x_6 \leq 5$$

$$3x_4 + x_5 - x_6 \leq 4$$

x_6 is a complicating variable:
if fixed to a given value, then the original problem decomposes to 3 subproblems

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Example 3

Identify whether the following problem is decomposable or not (if so, how)!

Single-node single-hour optimal power flow (OPF) problem (total cost minimization) with 3 conventional generators and a single inelastic load:

$$\underset{g_1, g_2, g_3}{\text{Minimize}} \quad 10g_1 + 25g_2 + 30g_3$$

subject to

$$0 \leq g_1 \leq 100$$

$$0 \leq g_2 \leq 150$$

$$0 \leq g_3 \leq 200$$

$$g_1 + g_2 + g_3 = 350$$

Example 3

Identify whether the following problem is decomposable or not (if so, how)!

Single-node single-hour optimal power flow (OPF) problem (total cost minimization) with 3 conventional generators and a single inelastic load:

$$\underset{g_1, g_2, g_3}{\text{Minimize}} \quad 10g_1 + 25g_2 + 30g_3$$

subject to

$$0 \leq g_1 \leq 100$$

$$0 \leq g_2 \leq 150$$

$$0 \leq g_3 \leq 200$$

$$g_1 + g_2 + g_3 = 350$$

→ **Complicating constraint:** if relaxed, then the original problem decomposes to 3 subproblems, one per generator

Example 4

Identify whether the following problem is decomposable or not (if so, how)!

Single-node single-hour optimal power flow (OPF) problem (social welfare maximization) with 3 conventional generators and a single elastic load:

$$\underset{g_1,g_2,g_3,d_1}{\text{Maximize}}\ 40d_1 - 10g_1 - 25g_2 - 30g_3$$

subject to

$$0 \leq g_1 \leq 100$$

$$0 \leq g_2 \leq 150$$

$$0 \leq g_3 \leq 200$$

$$0 \leq d_1 \leq 350$$

$$g_1 + g_2 + g_3 = d_1$$

Example 4

Identify whether the following problem is decomposable or not (if so, how)!

Single-node single-hour optimal power flow (OPF) problem (social welfare maximization) with 3 conventional generators and a single elastic load:

$$\underset{g_1,g_2,g_3,d_1}{\text{Maximize}} \; 40d_1 - 10g_1 - 25g_2 - 30g_3$$

subject to

$$0 \le g_1 \le 100$$

$$0 \le g_2 \le 150$$

$$0 \le g_3 \le 200$$

$$0 \le d_1 \le 350$$

$$g_1 + g_2 + g_3 = d_1$$

Complicating constraint:
if relaxed, then the original problem decomposes to 4 subproblems, one per agent (generator and demand)

Example 5

Identify whether the following problem is decomposable or not (if so, how)!

Single-node multi-hour (index: h) optimal power flow (OPF) problem (social welfare maximization) with 3 conventional generators and a single elastic load:

$$\underset{g_{h,1},g_{h,2},g_{h,3},d_{h,1}}{\text{Maximize}} \quad \sum_{h}\left[40d_{h,1} - 10g_{h,1} - 25g_{h,2} - 30g_{h,3}\right]$$

subject to

$$0 \le g_{h,1} \le 100 \quad \forall h$$

$$0 \le g_{h,2} \le 150 \quad \forall h$$

$$0 \le g_{h,3} \le 200 \quad \forall h$$

$$0 \le d_{h,1} \le D_{h,1} \quad \forall h$$

$$g_{h,1} + g_{h,2} + g_{h,3} = d_{h,1} \quad \forall h$$

Example 5

Identify whether the following problem is decomposable or not (if so, how)!

Single-node multi-hour (index: h) optimal power flow (OPF) problem (social welfare maximization) with 3 conventional generators and a single elastic load:

$$\underset{g_{h,1}, g_{h,2}, g_{h,3}, d_{h,1}}{\text{Maximize}} \sum_{h} [40 d_{h,1} - 10 g_{h,1} - 25 g_{h,2} - 30 g_{h,3}]$$

subject to

$$0 \le g_{h,1} \le 100 \quad \forall h$$

$$0 \le g_{h,2} \le 150 \quad \forall h$$

$$0 \le g_{h,3} \le 200 \quad \forall h$$

$$0 \le d_{h,1} \le D_{h,1} \quad \forall h$$

$$\boxed{g_{h,1} + g_{h,2} + g_{h,3} = d_{h,1} \quad \forall h}$$

Complicating constraints:
if relaxed, then the original problem decomposes to a set of subproblems, one per agent per hour

Example 6

Identify whether the following problem is decomposable or not (if so, how)!

Single-node multi-hour (index: h) optimal power flow (OPF) problem (social welfare maximization) with 3 conventional generators and a single elastic load (enforcing inter-temporal ramping constraints for one of generators):

$$\underset{g_{h,1},g_{h,2},g_{h,3},d_{h,1}}{\text{Maximize}} \quad \sum_{h}\left[40d_{h,1} - 10g_{h,1} - 25g_{h,2} - 30g_{h,3}\right]$$

subject to

$$0 \le g_{h,1} \le 100 \quad \forall h$$

$$0 \le g_{h,2} \le 150 \quad \forall h$$

$$0 \le g_{h,3} \le 200 \quad \forall h$$

$$0 \le d_{h,1} \le D_{h,1} \quad \forall h$$

$$-25 \le \left[g_{h,1} - g_{(h-1),1}\right] \le 25 \quad \forall h$$

$$g_{h,1} + g_{h,2} + g_{h,3} = d_{h,1} \quad \forall h$$

Example 6

Identify whether the following problem is decomposable or not (if so, how)!

Single-node multi-hour (index: h) optimal power flow (OPF) problem (social welfare maximization) with 3 conventional generators and a single elastic load (enforcing inter-temporal ramping constraints for one of generators):

$$\underset{g_{h,1},g_{h,2},g_{h,3},d_{h,1}}{\text{Maximize}} \sum_{h} \left[40d_{h,1} - 10g_{h,1} - 25g_{h,2} - 30g_{h,3}\right]$$

subject to

$$0 \leq g_{h,1} \leq 100 \quad \forall h$$

$$0 \leq g_{h,2} \leq 150 \quad \forall h$$

$$0 \leq g_{h,3} \leq 200 \quad \forall h$$

$$0 \leq d_{h,1} \leq D_{h,1} \quad \forall h$$

$$-25 \leq \left[g_{h,1} - g_{(h-1),1}\right] \leq 25 \quad \forall h$$

$$g_{h,1} + g_{h,2} + g_{h,3} = d_{h,1} \quad \forall h$$

Complicating constraints:
if relaxed, then the original problem decomposes to a set of subproblems, one per agent per hour

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Example 7

Identify whether the following problem is decomposable or not (if so, how)!

Single-node single-year (static) generation expansion problem (GEP), considering two existing generators (units 1 and 2) and one candidate generator (unit 3)

$$\underset{g_{h,1}, g_{h,2}, g_{h,3}, x_3}{\text{Minimize}} \quad 15000x_3 + \sum_h [10g_{h,1} + 25g_{h,2} + 30g_{h,3}]$$

subject to

$$0 \leq g_{h,1} \leq 100 \quad \forall h$$

$$0 \leq g_{h,2} \leq 150 \quad \forall h$$

$$0 \leq g_{h,3} \leq x_3 \quad \forall h$$

$$g_{h,1} + g_{h,2} + g_{h,3} = D_{h,1} \quad \forall h$$

$$x_3 \geq 0$$

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Example 7

Identify whether the following problem is decomposable or not (if so, how)!

Single-node single-year (static) generation expansion problem (GEP), considering two existing generators (units 1 and 2) and one candidate generator (unit 3)

$$\underset{g_{h,1},g_{h,2},g_{h,3},x_3}{\text{Minimize}} \quad 15000x_3 + \sum_h [10g_{h,1} + 25g_{h,2} + 30g_{h,3}]$$

subject to

$$0 \leq g_{h,1} \leq 100 \quad \forall h$$

$$0 \leq g_{h,2} \leq 150 \quad \forall h$$

$$0 \leq g_{h,3} \leq x_3 \quad \forall h$$

$$g_{h,1} + g_{h,2} + g_{h,3} = D_{h,1} \quad \forall h$$

$$x_3 \geq 0$$

x_3 is a complicating variable:
if fixed to a given value, then the original problem decomposes to a set of subproblems, one per hour

Reference

A. J. Conejo, E. Castillo, R. Minguez, and R. Garcia-Bertrand, *Decomposition Techniques in Mathematical Programming: Engineering and Science Applications*. Berlin, Germany: Springer, 2006.

DTU

Thanks for your attention!

Email: seykaz@elektro.dtu.dk

July 4th, 2016
Technical University of Denmark, Lyngby, Denmark

EES-UETP Course title

Decomposition techniques for optimization problems with complicating constraints

Jalal Kazempour
Technical University of Denmark (DTU)

Learning Objectives

After this session the participants are expected to be able to:

- Explain the functioning of Lagrangian relaxation (LR), augmented Lagrangian relaxation (ALR), and alternating direction method of multipliers (ADMM)

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Decomposition Techniques

Applicable to optimization problems with complicating constraints

- Lagrangian relaxation (LR)

  In the literature, this technique has been also known as *standard* or *conventional* LR!

- Augmented Lagrangian relaxation (ALR)
  - ❑ Auxiliary problem principle (APP)
  - ❑ Alternating direction method of multipliers (ADMM)

- Dantzig-Wolfe decomposition (DWD)

- …

Decomposition Techniques

Applicable to optimization problems with complicating constraints

- Lagrangian relaxation (LR)

  In the literature, this technique has been also known as ***standard*** or ***conventional*** LR!

- Augmented Lagrangian relaxation (ALR)
  - ❑ Auxiliary problem principle (APP)
  - ❑ Alternating direction method of multipliers (ADMM)

- Dantzig-Wolfe decomposition (DWD)
- …

Will not be covered in this course!

Optimization problems with complicating constraints

Some examples related to power systems

- **Single-node optimal power flow (OPF) problem (investigated in the previous lecture)**
  - ✓ **Complicating constraints**: balance equalities and ramping limits of generators
  - ✓ If relaxed, original problem decomposes by agent (and hour)

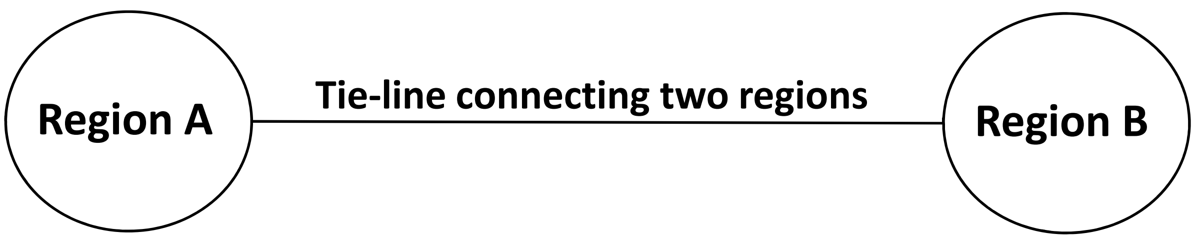
Optimization problems with complicating constraints

Some examples related to power systems

- **Single-node optimal power flow (OPF) problem (investigated in the previous lecture)**

  ✓ **Complicating constraints**: balance equalities and ramping limits of generators
  ✓ If relaxed, original problem decomposes by agent (and hour)

- **Multi-regional OPF (or unit commitment) problem, e.g., in case of pan-European electricity market**



  ✓ **Complicating constraints**: tie-line constraints (power flow, and tie-line capacity)
  ✓ If relaxed, original problem decomposes by region. This way, operator of each region only solves its own OPF problem (so-called *distributed* OPF problem)

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Lagrangian Relaxation (LR)

Background

- The theory of LR (and also ALR) was firstly developed for problems with continuous variables, and functions (objective function and constraints) with first derivatives continuous.

- However, the theory has been used in problems with binary variables (like unit commitment problems) with success.

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Lagrangian Relaxation (LR)

Background

- LR works efficiently if the number of complicating constraints is relatively low, and it is OK to have binary variables in the formulation.

- LR was extensively used in the 90's to solve unit commitment problems (complicating constrains are just balance constraints and ramping constraints).

Key point

In case of LR:

In addition to convexity, the objective function of the original problem (not decomposed problem) needs to be smooth (continuous first derivatives), e.g., quadratic. If this objective function is linear, the LR procedure does not converge!

- Alternative solution technique for problems with linear objective function is ALR.

Lagrangian Relaxation (LR)

Background

For unit commitment (and also OPF) problems:

- LR (for problems with quadratic objective function)
- ALR (for problems with either quadratic or linear objective function)

Both have been extensively and very successfully used in the literature, though unit commitment problem is fully non-convex (due to binary variables).

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Lagrangian Relaxation (LR)

Mathematical procedure

Original (non-decomposed) problem:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i)$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$

$$h_i(x_i) \leq B_i \qquad \forall i$$

$$\sum_{i=1}^{I} c_i(x_i) = M$$

$$\sum_{i=1}^{I} d_i(x_i) \leq N$$

Original (non-decomposed) problem:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i)$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$

$$h_i(x_i) \leq B_i \qquad \forall i$$

Complicating constraints:

$$\sum_{i=1}^{I} c_i(x_i) = M$$

$$\sum_{i=1}^{I} d_i(x_i) \leq N$$

Original (non-decomposed) problem:

$$\underset{x_i}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i)$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$

$$h_i(x_i) \leq B_i \qquad \forall i$$

Complicating constraints:

$$\sum_{i=1}^{I} c_i(x_i) = M \qquad (\lambda)$$

$$\sum_{i=1}^{I} d_i(x_i) \leq N \qquad (\mu)$$

Dual variables (Lagrangian multipliers)

Lagrangian Relaxation (LR)

Mathematical procedure

In the optimal point, the original problem is equivalent to:

$$\underset{x_i,\lambda,\mu}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i) + \lambda \left[M - \sum_{i=1}^{I} c_i(x_i) \right] + \mu \left[N - \sum_{i=1}^{I} d_i(x_i) \right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$
$$h_i(x_i) \leq B_i \qquad \forall i$$

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Lagrangian Relaxation (LR)

Mathematical procedure

In the optimal point, the original problem is equivalent to:

$$\underset{x_i,\lambda,\mu}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i) + \lambda \left[M - \sum_{i=1}^{I} c_i(x_i) \right] + \mu \left[N - \sum_{i=1}^{I} d_i(x_i) \right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$

$$h_i(x_i) \leq B_i \qquad \forall i$$

Is the original problem decomposable now?

In the optimal point, the original problem is equivalent to:

$$\underset{x_i,\lambda,\mu}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i) + \lambda \left[M - \sum_{i=1}^{I} c_i(x_i)\right] + \mu \left[N - \sum_{i=1}^{I} d_i(x_i)\right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$

$$h_i(x_i) \leq B_i \qquad \forall i$$

Is the original problem decomposable now? Not yet!

Lagrangian Relaxation (LR)

Mathematical procedure

In the optimal point, the original problem is equivalent to:

$$\underset{x_i,\lambda,\mu}{\text{Minimize}} \sum_{i=1}^{I} f_i(x_i) + \lambda \left[M - \sum_{i=1}^{I} c_i(x_i)\right] + \mu \left[N - \sum_{i=1}^{I} d_i(x_i)\right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$
$$h_i(x_i) \leq B_i \qquad \forall i$$

Is the original problem decomposable now? Not yet!

- Let's **<u>relax</u>** the equivalent problem above by **<u>fixing dual variables</u> (λ and μ)** to given values, i.e., $\bar{\lambda}$ **and** $\bar{\mu}$.

Lagrangian Relaxation (LR)

Mathematical procedure

In the optimal point, the original problem is equivalent to:

$$\underset{x_i}{\text{Minimize}} \quad \sum_{i=1}^{I} f_i(x_i) + \color{red}{\bar{\boldsymbol{\lambda}}} \left[M - \sum_{i=1}^{I} c_i(x_i) \right] + \color{red}{\bar{\boldsymbol{\mu}}} \left[N - \sum_{i=1}^{I} d_i(x_i) \right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$
$$h_i(x_i) \leq B_i \qquad \forall i$$

Is the original problem decomposable now?

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In the optimal point, the original problem is equivalent to:

$$\underset{x_i}{\text{Minimize}} \ \sum_{i=1}^{I} f_i(x_i) + \bar{\lambda}\left[M - \sum_{i=1}^{I} c_i(x_i)\right] + \bar{\mu}\left[N - \sum_{i=1}^{I} d_i(x_i)\right]$$

Subject to

$$g_i(x_i) = A_i \qquad \forall i$$
$$h_i(x_i) \leq B_i \qquad \forall i$$

Is the original problem decomposable now? Yes, one per i:

$$\left\{\underset{x_i}{\text{Minimize}} \ f_i(x_i) + \bar{\lambda}\, c_i(x_i) + \bar{\mu}\, d_i(x_i)\right.$$

Subject to

$$\left.\begin{array}{l} g_i(x_i) = A_i \\ h_i(x_i) \leq B_i \end{array}\right\} \forall i$$

Lagrangian Relaxation (LR)

Mathematical procedure

LR is an iterative approach with a systematic way to update the values of fixed dual variables ($\bar{\lambda}$ and $\bar{\mu}$) in each iteration.

Available techniques in the literature to update $\bar{\lambda}$ and $\bar{\mu}$:

1. Subgradient method
2. Cutting plane method
3. Bundle method
4. Trust region method
5. …

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Lagrangian Relaxation (LR)

Mathematical procedure

LR is an iterative approach with a systematic way to update the values of fixed dual variables ($\bar{\lambda}$ and $\bar{\mu}$) in each iteration.

Available techniques in the literature to update $\bar{\lambda}$ and $\bar{\mu}$:

1. Subgradient method
2. Cutting plane method
3. Bundle method
4. Trust region method
5. …

Will not be covered in this course!

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Lagrangian Relaxation (LR)

Numerical example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}}\ x^2 + y^2$$

Subject to $-x - y \leq -4 \qquad (\mu)$

Note: Objective function includes quadratic terms, so LR works!

Lagrangian Relaxation (LR)

Numerical example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}}\ x^2 + y^2$$

Subject to $-x - y \leq -4 \qquad (\mu)$

Note: Objective function includes quadratic terms, so LR works!

Subproblem 1:

$$\underset{x\geq 0}{\text{Minimize}}\ x^2 - \bar{\mu}x$$

Subproblem 2:

$$\underset{y\geq 0}{\text{Minimize}}\ y^2 - \bar{\mu}y$$

Lagrangian Relaxation (LR)

Numerical example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

$$\text{Subject to} \quad -x - y \leq -4 \qquad (\mu)$$

Note: Objective function includes quadratic terms, so LR works!

Subproblem 1:

$$\underset{x\geq 0}{\text{Minimize}} \; x^2 - \bar{\mu}x$$

Subproblem 2:

$$\underset{y\geq 0}{\text{Minimize}} \; y^2 - \bar{\mu}y$$

Updating fixed dual variable ($\bar{\mu}$) using subgradient method:

- Solve subproblems 1 and 2 in iteration υ, and obtain the values $x^{(\upsilon)}$ and $y^{(\upsilon)}$
- $\bar{\mu}^{(\upsilon+1)} \leftarrow \bar{\mu}^{(\upsilon)} + \frac{1}{a+b\upsilon} \frac{-x^{(\upsilon)}-y^{(\upsilon)}+4}{\left|-x^{(\upsilon)}-y^{(\upsilon)}+4\right|}$
- a and b are positive constants, e.g., $a = 1$ and $b = 0.1$.

Lagrangian Relaxation (LR)

Numerical example

Algorithm:

- **Step 0: Initialization**

  Set $\upsilon = 1$ and $\bar{\mu}^{(1)} = \bar{\mu}^{\text{initial}}$

- **Step 1: Solve subproblems 1 and 2, and obtain $\boldsymbol{x}^{(\boldsymbol{\upsilon})}$ and $\boldsymbol{y}^{(\boldsymbol{\upsilon})}$**

- **Step 2: Update fixed dual variable, i.e., $\overline{\boldsymbol{\mu}}^{(\boldsymbol{\upsilon}+\mathbf{1})}$**

- **Step 3: Convergence check**

  If $\frac{\|\overline{\mu}^{(\upsilon+1)}-\overline{\mu}^{(\upsilon-1)}\|}{\|\overline{\mu}^{(\upsilon)}\|} \leq \epsilon$, then the optimal solution with a level of accuracy ϵ is obtained, otherwise $\upsilon \leftarrow \upsilon + 1$ and go Step 1

Lagrangian Relaxation (LR)

Numerical example

Exercise:

The printed version of GAMS code of the LR example is available on your table. Please check it in the next 15 minutes, then explain it to your neighbors around the table.

My colleagues and I will answer your questions!

This code has been prepared by Lejla Halilbasic and Christos Ordoudis.

Augmented Lagrangian Relaxation (ALR)

Functioning procedure

Recall:

ALR works for problems with either quadratic objective function (like LR) or linear one (unlike LR)

Main difference of ALR with respect to LR:

An additional penalty term within the subproblems

Augmented Lagrangian Relaxation (ALR)

Numerical Example

$$\underset{x \geq 0, y \geq 0}{\text{Minimize}} \; x^2 + y^2$$

Subject to $-x - y = -4 \;\; (\lambda)$

Augmented Lagrangian Relaxation (ALR)

Numerical Example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

Subject to $\; -x - y = -4 \quad (\lambda)$

Additional penalty term with respect to LR,
γ is a positive constant

Equivalent to:

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2 + \lambda(-x - y + 4) + \frac{\gamma}{2} \|-x - y + 4\|^2$$

Augmented Lagrangian Relaxation (ALR)

Numerical Example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

Subject to $\quad -x - y = -4 \quad (\lambda)$

Additional penalty term with respect to LR, γ is a positive constant

Equivalent to:

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2 + \lambda(-x - y + 4) + \frac{\gamma}{2}\|-x - y + 4\|^2$$

Question:

- **Similar to LR, assume dual variable λ is fixed to given value $\bar{\lambda}$. Is the problem above decomposable for given $\bar{\lambda}$?**

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2$$

$$\text{Subject to} \quad -x - y = -4 \quad (\lambda)$$

Additional penalty term with respect to LR, γ is a positive constant

Equivalent to:

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}} \; x^2 + y^2 + \lambda(-x - y + 4) + \frac{\gamma}{2}\|-x - y + 4\|^2$$

Question:

- **Similar to LR, assume dual variable λ is fixed to given value $\bar{\lambda}$. Is the problem above decomposable for given $\bar{\lambda}$? No, due to product of x and y in the penalty term!**

Augmented Lagrangian Relaxation (ALR)

Numerical Example

Available alternatives to solve ALR:

- Auxiliary problem principle (APP)
- Alternating direction method of multipliers (ADMM)

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

Available alternatives to solve ALR:

- Auxiliary problem principle (APP): *will not be covered in this course*
- **Alternating direction method of multipliers (ADMM)**

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Augmented Lagrangian Relaxation (ALR)

Numerical Example

Available alternatives to solve ALR:

- Auxiliary problem principle (APP): *will not be covered in this course*
- **Alternating direction method of multipliers (ADMM)**

Note:

ADMM directly fixes variables to their values in the previous iteration, and decomposes the ALR to subproblems.

Augmented Lagrangian Relaxation (ALR)

Numerical Example

Available alternatives to solve ALR:

- Auxiliary problem principle (APP): *will not be covered in this course*
- **Alternating direction method of multipliers (ADMM)**

Note:

ADMM directly fixes variables to their values in the previous iteration, and decomposes the ALR to subproblems.

$$\underset{x\geq 0, y\geq 0}{\text{Minimize}}\ x^2 + y^2 + \lambda(-x-y+4) + \frac{\gamma}{2}\|-x-y+4\|^2$$

The problem above in iteration υ can be decomposed to two subproblems:

$$\begin{cases} \underset{x^{(\upsilon)}\geq 0}{\text{Minimize}}\ x^{2(\upsilon)} + \lambda^{(\upsilon-1)}(-x^{(\upsilon)}+2) + \frac{\gamma}{2}\left\|-x^{(\upsilon)} - y^{(\upsilon-1)} + 4\right\|^2 \\ \underset{y^{(\upsilon)}\geq 0}{\text{Minimize}}\ y^{2(\upsilon)} + \lambda^{(\upsilon-1)}(-y^{(\upsilon)}+2) + \frac{\gamma}{2}\left\|-y^{(\upsilon)} - x^{(\upsilon-1)} + 4\right\|^2 \end{cases}$$

where $\lambda^{(\upsilon)} \leftarrow \lambda^{(\upsilon-1)} + \gamma\left(-x^{(\upsilon)} - y^{(\upsilon)} + 4\right)$

Augmented Lagrangian Relaxation (ALR)

Numerical Example

Exercise:

The printed version of GAMS code of the ALR/ADMM example is available on your table. Please check it in the next 15 minutes, then explain it to your neighbors around the table.

My colleagues and I will answer your questions!

This code has been prepared by Lejla Halilbasic and Christos Ordoudis.

References

- A. J. Conejo, E. Castillo, R. Minguez, and R. Garcia-Bertrand, *Decomposition Techniques in Mathematical Programming: Engineering and Science Applications*. Berlin, Germany: Springer, 2006.

- S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein, "Distributed optimization and statistical learning via the alternating direction method of multipliers," Foundations and Trends in Machine Learning, vol. 3, no. 1, pp. 1-122, Jan. 2011.

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- Q. Peng and S. H. Low, "Distributed optimal power flow algorithm for radial networks, I: Balanced single phase case," IEEE Trans. Smart Grid, to be published, 2016.

Thanks for your attention!

Email: seykaz@elektro.dtu.dk

July 5th, 2016
Technical University of Denmark, Lyngby, Denmark

EES-UETP Course title

Decomposition techniques for optimization problems with complicating variables

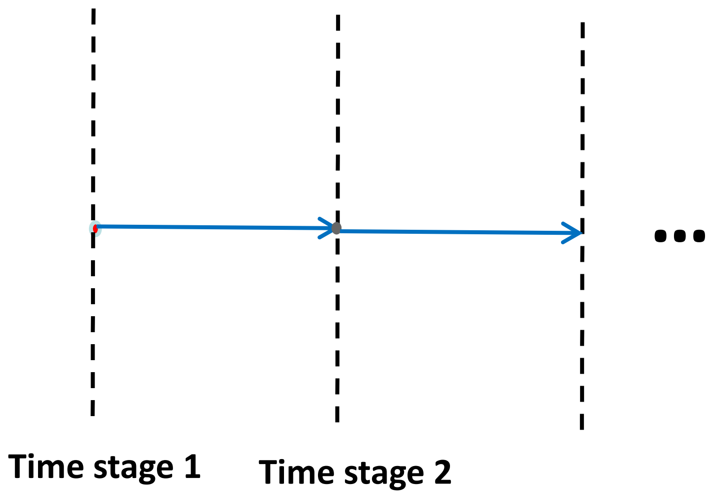
Jalal Kazempour
Technical University of Denmark (DTU)

Learning Objectives

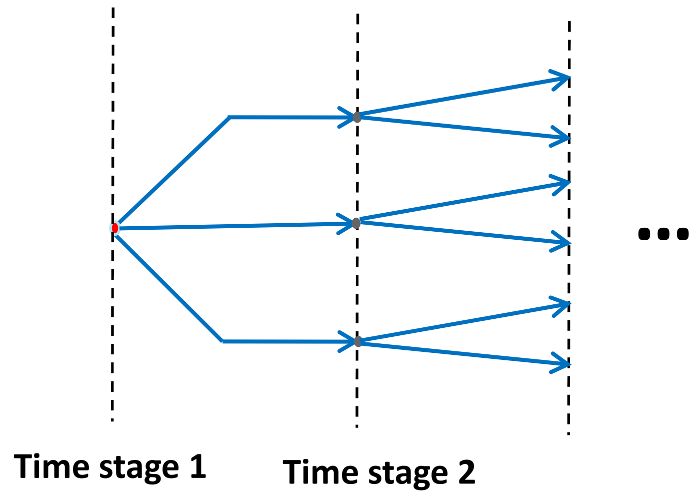
After this session the participants are expected to be able to:

- Explain the functioning of Benders' decomposition
- Explain the application of Benders' decomposition to multi-stage stochastic problems

Multi-stage Problems



Deterministic multi-stage problem
(e.g., deterministic unit commitment problem)



Stochastic multi-stage problem
(e.g., stochastic unit commitment problem)

Two-stage Deterministic Problem

$$\min_{x_1, x_2} \quad c_1 x_1 + c_2\, x_2$$

subject to

$$A_1 x_1 \geq b_1$$

$$E_1 x_1 + A_2 x_2 \geq b_2$$

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Two-stage Deterministic Problem

First-stage cost

$$\min_{x_1,x_2} \; \underbrace{c_1 x_1}_{\text{First-stage cost}} + \underbrace{c_2\, x_2}_{\text{Second-stage cost}}$$

Second-stage cost

subject to

$$A_1 x_1 \geq b_1$$ ← First-stage constraint

$$\underbrace{E_1 x_1 + A_2 x_2 \geq b_2}_{\text{Linking constraint}}$$

Linking constraint

Two-stage Deterministic Problem

$$\min_{x_1,x_2} \; c_1 x_1 + c_2\, x_2$$

subject to

$$A_1 x_1 \geq b_1$$

$$E_1 x_1 + A_2 x_2 \geq b_2$$

First-stage problem:

$$\min_{x_1} \; c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1$$

$\alpha_1(x_1)$: the second-stage cost as a function of the first-stage decisions x_1 (future cost function)

Two-stage Deterministic Problem

First-stage problem:

$$\min_{x_1} \; c_1 x_1 + \alpha_1(x_1)$$
$$\text{subject to}$$
$$A_1 x_1 \geq b_1$$

Note: x_1 appears in the second-stage problem!

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$
$$\text{subject to}$$
$$A_2 x_2 \geq b_2 - E_1 x_1$$

$\alpha_1(x_1)$: the second-stage cost as a function of the first-stage decisions x_1 (future cost function)

Two-stage Deterministic Problem

One potential solution approach

First-stage problem:

$$\min_{x_1} \; c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1$$

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Two-stage Deterministic Problem

One potential solution approach

- **Step 1) Discretize** x_1 into a set of trial values $\{\hat{x}_{1i}, i = 1, \dots, n\}$
- **Step 2)** Solve the second-stage problem for each of the trial values
- **Step 3)** Construct future cost function $\alpha_1(x_1)$. Intermediate values of $\alpha_1(x_1)$ are obtained by interpolation from the neighboring discretized values.
- **Step 4)** Solve the first-stage problem using the future cost function constructed.

First-stage problem:

$$\min_{x_1} \; c_1 x_1 + \alpha_1(x_1)$$

$$\text{subject to}$$

$$A_1 x_1 \geq b_1$$

Second-stage problem:

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

$$\text{subject to}$$

$$A_2 x_2 \geq b_2 - E_1 x_1$$

Two-stage Deterministic Problem

One potential solution approach

- **Step 1)** **Discretize** x_1 into a set of trial values $\{\hat{x}_{1i}, i = 1, \ldots, n\}$
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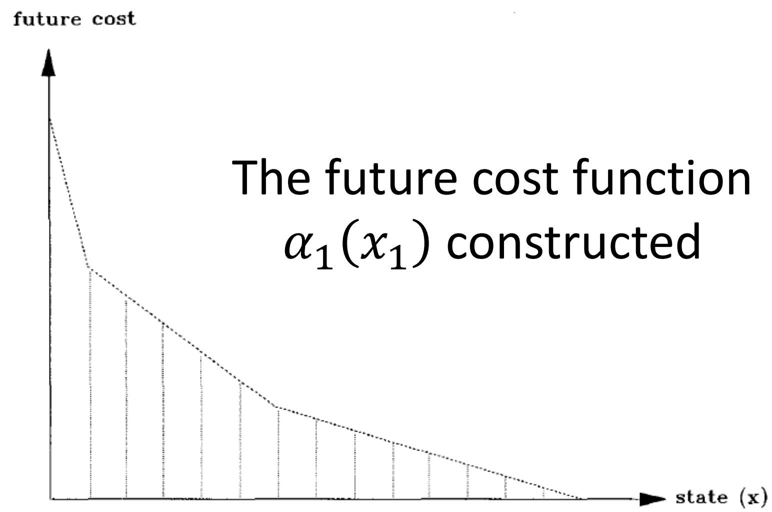

The future cost function $\alpha_1(x_1)$ constructed

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Two-stage Deterministic Problem

One potential solution approach

- **Step 1)** **Discretize** x_1 into a set of trial values $\{\hat{x}_{1i}, i = 1, \ldots, n\}$
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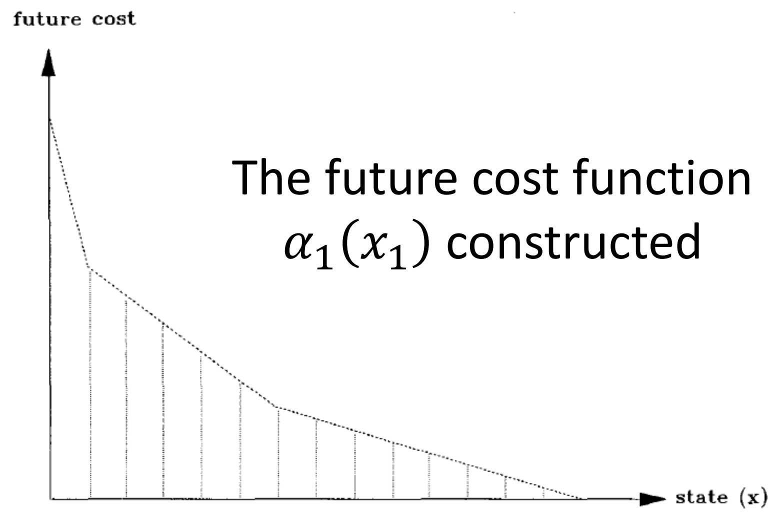

The future cost function $\alpha_1(x_1)$ constructed

What is the name of this technique?

Two-stage Deterministic Problem

One potential solution approach

- **Step 1)** **Discretize** x_1 into a set of trial values $\{\hat{x}_{1i}, i = 1, \ldots, n\}$
- **Step 2)** Solve the second-stage problem for each of the trial values
- **Step 3)** Construct future cost function $\alpha_1(x_1)$. Intermediate values of $\alpha_1(x_1)$ are obtained by interpolation from the neighboring discretized values.
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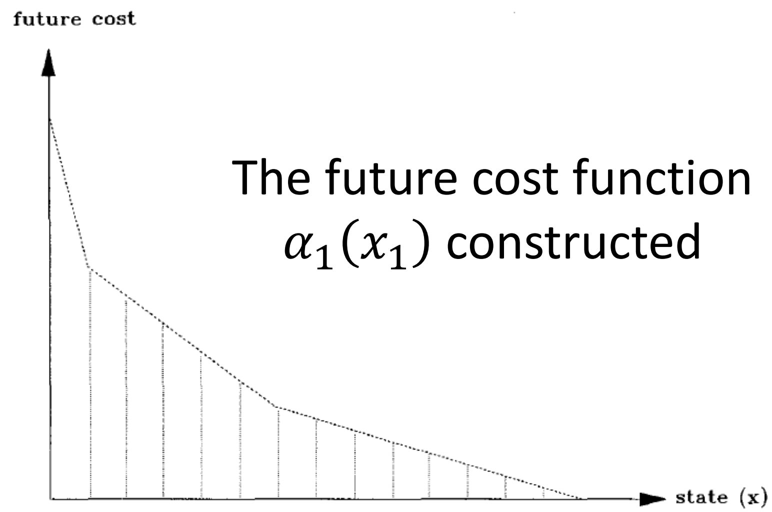


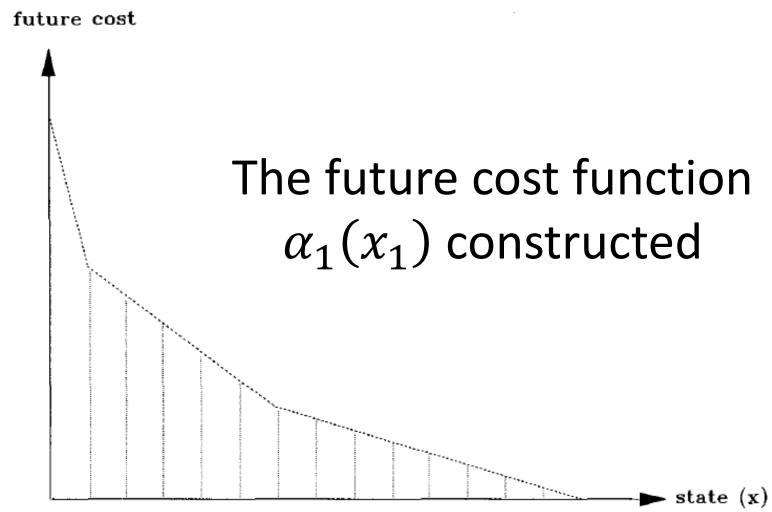
The future cost function $\alpha_1(x_1)$ constructed

What is the name of this technique?

Dynamic programming (DP)

Two-stage Deterministic Problem

One potential solution approach



The future cost function $\alpha_1(x_1)$ constructed

What is the main drawback of dynamic programming (DP)?

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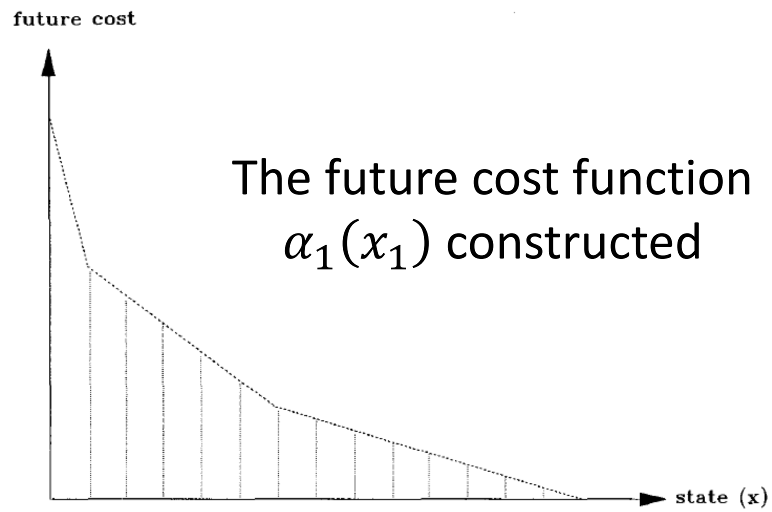
Two-stage Deterministic Problem

One potential solution approach



The future cost function $\alpha_1(x_1)$ constructed

What is the main drawback of dynamic programming (DP)?

DP needs to discretize the decision variables x_1, which results in computational issues!

For example, 10 decision variables and 4 discretized value for each variable leads to 4^{10} discrete values!

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Two-stage Deterministic Problem

Alternative solution approach

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Two-stage Deterministic Problem

Alternative solution approach

Dual dynamic programming (DDP) instead of DP!

Advantage:

To approximate the future cost function $\alpha_1(x_1)$ by analytical functions rather than a set of discrete values!

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Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

$$\text{subject to}$$

$$A_2 x_2 \geq b_2 - E_1 x_1 \quad : \; \pi$$

Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

$$\text{subject to}$$

$$A_2 x_2 \geq b_2 - E_1 x_1 \quad : \; \pi$$

Dual of the second-stage problem:

$$\max_{\pi} \; \pi(b_2 - E_1 x_1)$$

$$\text{subject to}$$

$$\pi A_2 \geq c_2$$

Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

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$$\max_{\pi} \; \pi(b_2 - E_1 x_1)$$

subject to

$$\pi A_2 \geq c_2$$

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

In the optimal solution (strong duality theorem)

Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \min_{x_2} \; c_2 x_2$$

subject to

$$A_2 x_2 \geq b_2 - E_1 x_1 \quad : \; \pi$$

Dual of the second-stage problem:

$$\max_{\pi} \; \pi(b_2 - E_1 x_1)$$

subject to

$$\pi A_2 \geq c_2$$

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

In the optimal solution (strong duality theorem)

Interpretation: there is a linear relation between x_1 and the future cost function $\alpha_1(x_1)$ if the sensitivity π^* is known!

Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

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Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π^*. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha,\, x_1} \ \alpha$$

subject to

$$\alpha \ \geq \ \pi^1(b_2 - E_1 x_1)$$

$$\alpha \ \geq \ \pi^2(b_2 - E_1 x_1)$$

.

.

.

$$\alpha \ \geq \ \pi^n(b_2 - E_1 x_1)$$

Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π^*. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha,\, x_1} \; \alpha$$

$$\text{subject to}$$

$$\alpha \geq \pi^1(b_2 - E_1 x_1)$$

$$\alpha \geq \pi^2(b_2 - E_1 x_1)$$

$$\vdots$$

$$\alpha \geq \pi^n(b_2 - E_1 x_1)$$

Let's interpret this optimization problem!

Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π^*. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha,\, x_1} \alpha$$

subject to

$$\alpha \geq \pi^1(b_2 - E_1 x_1)$$

$$\alpha \geq \pi^2(b_2 - E_1 x_1)$$

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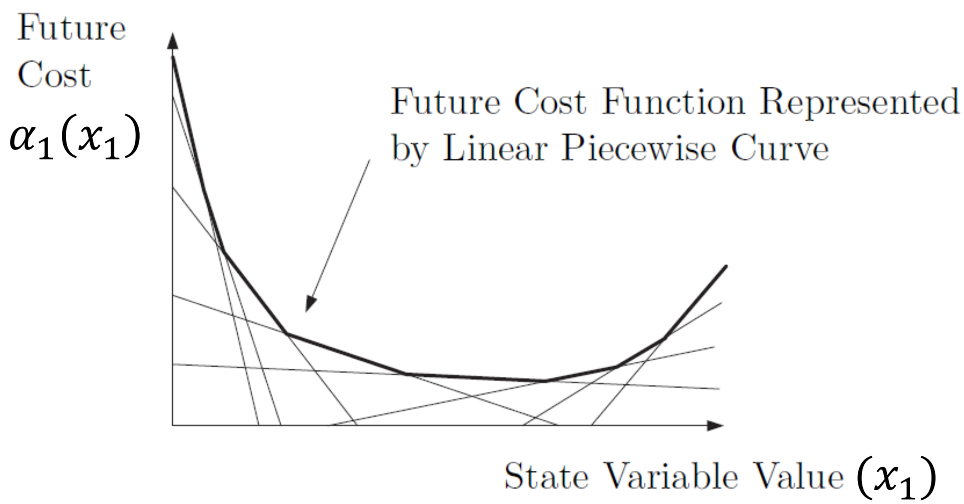
.

$$\alpha \geq \pi^n(b_2 - E_1 x_1)$$



Note: this means that we can construct a piecewise linear function for $\alpha_1(x_1)$ problem (analytically but approximately) without need to discretize x_1!

Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π^*. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha,\, x_1} \ \alpha$$

subject to

$$\alpha \ \geq \ \pi^1(b_2 - E_1 x_1)$$

$$\alpha \ \geq \ \pi^2(b_2 - E_1 x_1)$$

.

.

.

$$\alpha \ \geq \ \pi^n(b_2 - E_1 x_1)$$

Recall the first-stage problem:

$$\min_{x_1} \ c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Two-stage Deterministic Problem

DDP functioning procedure

$$\alpha_1(x_1) = \pi^*(b_2 - E_1 x_1)$$

Assume π^1, π^2, …, π^n are possible values for π^*. Then, $\alpha_1(x_1)$ can be characterized as follows:

$$\alpha_1(x_1) = \min_{\alpha,\, x_1} \; \alpha$$

subject to

$$\alpha \;\geq\; \pi^1(b_2 - E_1 x_1)$$

$$\alpha \;\geq\; \pi^2(b_2 - E_1 x_1)$$

.

.

.

$$\alpha \;\geq\; \pi^n(b_2 - E_1 x_1)$$

Recall the first-stage problem:

$$\min_{x_1} \; c_1 x_1 + \alpha_1(x_1)$$

subject to

$$A_1 x_1 \geq b_1$$

Let's merge them!

Two-stage Deterministic Problem

DDP functioning procedure

Updated first-stage problem including the piecewise linear function $\alpha_1(x_1)$

$$\min_{\alpha,\, x_1} \; c_1 x_1 + \alpha$$

subject to

$$A_1 x_1 \geq b_1$$

$$\alpha \; \geq \; \pi^1 (b_2 - E_1 x_1)$$

$$\alpha \; \geq \; \pi^2 (b_2 - E_1 x_1)$$

.

.

.

$$\alpha \; \geq \; \pi^n (b_2 - E_1 x_1)$$

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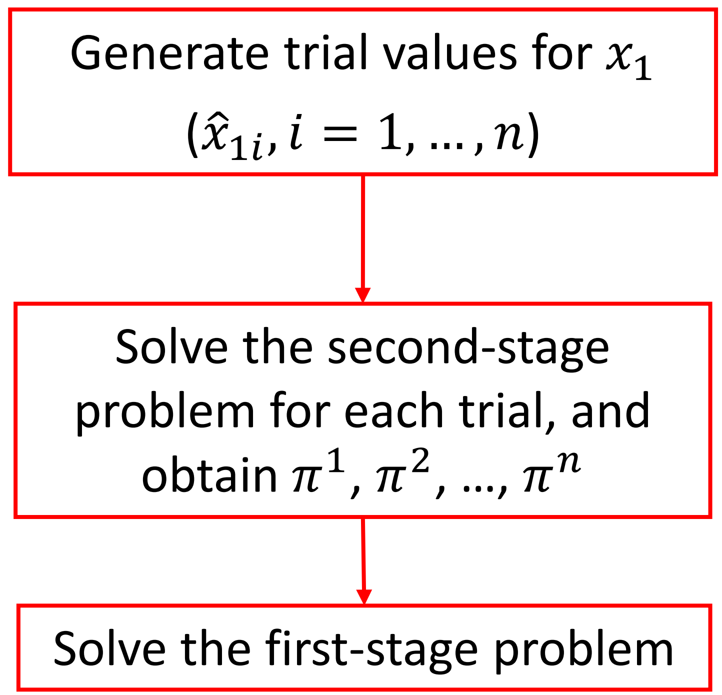
How to generate possible values for π^*, i.e., π^1, π^2, …, π^n ?

Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 1:



Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 1:

Generate trial values for x_1 $(\hat{x}_{1i}, i = 1, \ldots, n)$

↓

Solve the second-stage problem for each trial, and obtain $\pi^1, \pi^2, \ldots, \pi^n$

↓

Solve the first-stage problem

Do you recommend this option?

Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

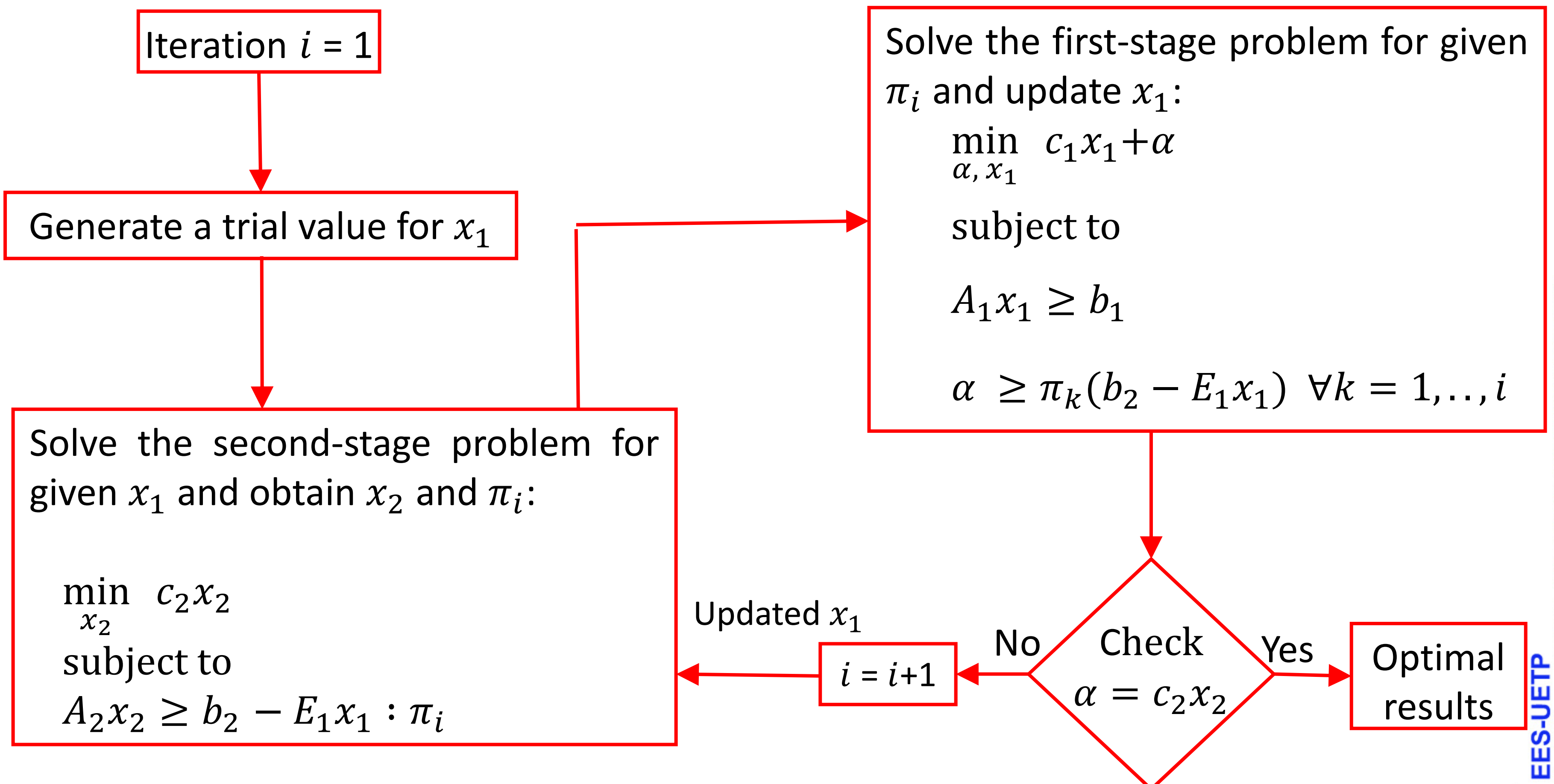
Option 2 (<u>systematic iterative</u> approach):

Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 2 (systematic iterative approach):

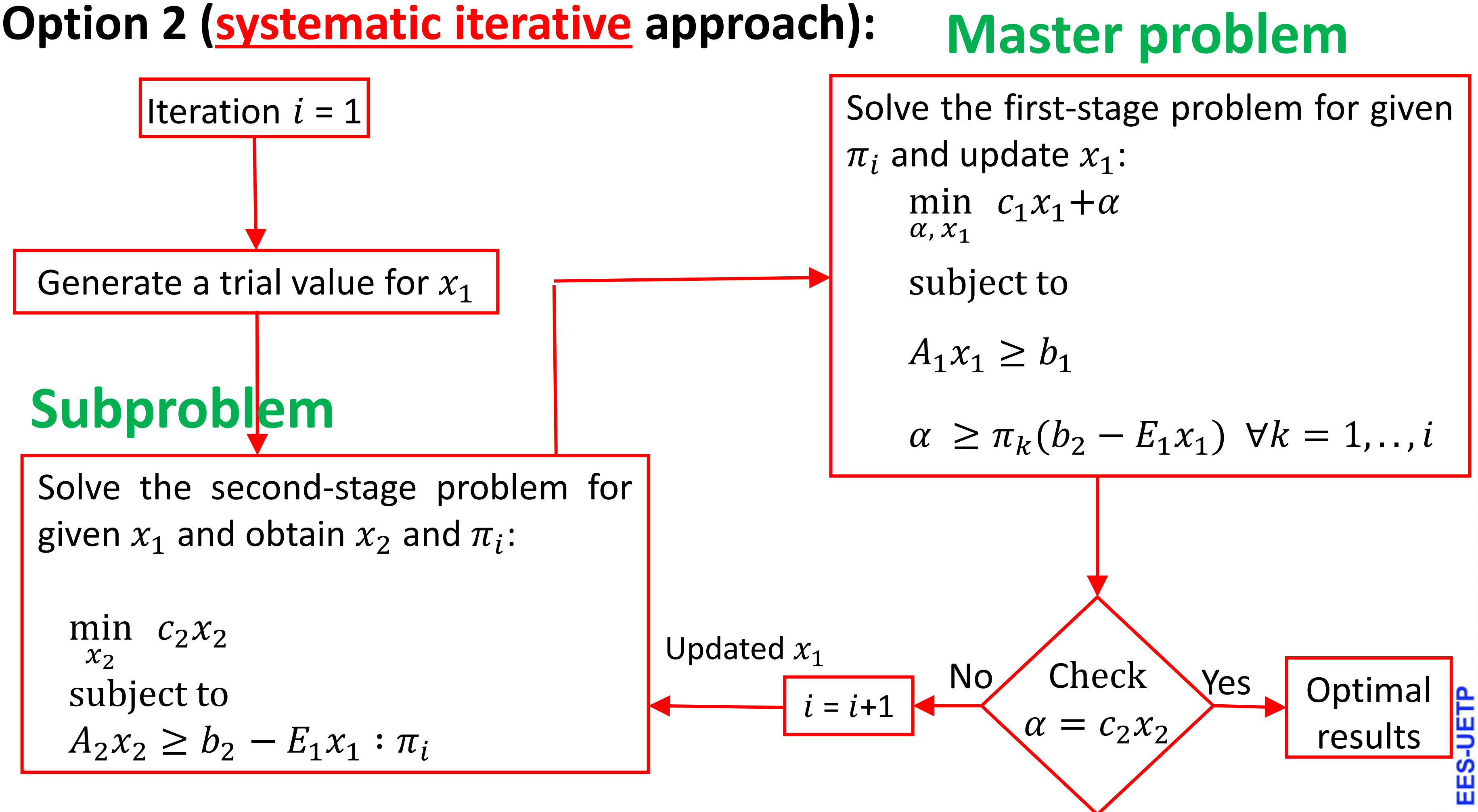


Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \ldots, \pi^n$?

Option 2 (systematic iterative approach):



Two-stage Deterministic Problem

DDP functioning procedure

How to generate possible values for π^*, i.e., $\pi^1, \pi^2, \dots, \pi^n$?

Option 2 (systematic iterative approach):

This approach is indeed Benders' decomposition!

Important note:

We can guarantee obtaining the global optimal solution by Benders' decomposition, **if** the objective function of the original (non-decomposed) problem is **convex** with respect to the complicating variable!

Two-stage Deterministic Problem

Simple Example

$$\underset{x,y}{\text{minimize}} \qquad z = -y - x/4$$

$$\begin{aligned}
y \quad & -x && \leq 5 \\
y \quad & -\tfrac{1}{2}x && \leq \tfrac{15}{2} \\
y \quad & +\tfrac{1}{2}x && \leq \tfrac{35}{2} \\
-y \quad & +x && \leq 10 \\
& 0 && \leq x \leq 16 \\
& y && \geq 0\,.
\end{aligned}$$

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Two-stage Deterministic Problem

Simple Example

$$\underset{x,y}{\text{minimize}} \quad z = -y - x/4$$

$$\begin{aligned} y \quad -x &\leq 5 \\ y \quad -\tfrac{1}{2}x &\leq \tfrac{15}{2} \\ y \quad +\tfrac{1}{2}x &\leq \tfrac{35}{2} \\ -y \quad +x &\leq 10 \\ 0 \leq x &\leq 16 \\ y &\geq 0\,. \end{aligned}$$

Let's consider *y* as the complicating variable!

Two-stage Deterministic Problem

Simple Example

i : current Benders' iteration
k: set of previous iterations

Subproblem:

$$\underset{y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} \leq 5 + x^{\text{fixed }(i)} \quad : \pi^{(i)}$$

$$y^{(i)} \leq \frac{15}{2} + \frac{x^{\text{fixed }(i)}}{2} \quad : \mu^{(i)}$$

$$y^{(i)} \leq \frac{35}{2} - \frac{x^{\text{fixed }(i)}}{2} \quad : \sigma^{(i)}$$

$$y^{(i)} \leq 10 - x^{\text{fixed }(i)} \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

Master problem:

$$\underset{x^{(i)},\, \alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

subject to

$$0 \leq x^{(i)} \leq 16$$

$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

$$\alpha^{(i)} \geq \pi^{(k)}\left[5 + x^{(i)}\right] + \mu^{(k)}\left[\frac{15}{2} + \frac{x^{(i)}}{2}\right] + \sigma^{(k)}\left[\frac{35}{2} - \frac{x^{(i)}}{2}\right]$$
$$+ \gamma^{(k)}\left[10 - x^{(i)}\right] \quad \forall k = 1, .., i-1$$

Two-stage Deterministic Problem

Simple Example

i : current Benders' iteration
k: set of previous iterations

Subproblem:

$$\underset{y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} \leq 5 + x^{\text{fixed }(i)} \quad : \pi^{(i)}$$

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$$y^{(i)} \leq 10 - x^{\text{fixed }(i)} \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

Note: In subproblem, symbols following colon are dual variables (sensitivities).

Master problem:

$$\underset{x^{(i)},\, \alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

subject to

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$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

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Two-stage Deterministic Problem

Simple Example

Subproblem:

$$\underset{y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} \leq 5 + x^{\text{fixed }(i)} \quad : \pi^{(i)}$$

$$y^{(i)} \leq \frac{15}{2} + \frac{x^{\text{fixed }(i)}}{2} \quad : \mu^{(i)}$$

$$y^{(i)} \leq \frac{35}{2} - \frac{x^{\text{fixed }(i)}}{2} \quad : \sigma^{(i)}$$

$$y^{(i)} \leq 10 - x^{\text{fixed }(i)} \quad : \gamma^{(i)}$$

$$y^{(i)} \geq 0$$

i : current Benders' iteration

k: set of previous iterations

Note: In subproblem, symbols following colon are dual variables (sensitivities).

Note: The last constraint of master problem generate "cuts".

Master problem:

$$\underset{x^{(i)},\, \alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

subject to

$$0 \leq x^{(i)} \leq 16$$

$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

$$\alpha^{(i)} \geq \pi^{(k)}\left[5 + x^{(i)}\right] + \mu^{(k)}\left[\frac{15}{2} + \frac{x^{(i)}}{2}\right] + \sigma^{(k)}\left[\frac{35}{2} - \frac{x^{(i)}}{2}\right] + \gamma^{(k)}\left[10 - x^{(i)}\right] \quad \forall k = 1, .., i-1$$

Algorithm:

- **Step 0: Initialization**

  Set $i = 1, x^{\text{fixed (1)}} = x^{\text{initial}}$, and lower bound (LB) = $-\infty$

- **Step 1: Solve subproblem(s):** obtain the values of all dual variables (sensitivities), and the value of objective function, which is upper bound (UB)

- **Step 2: Convergence check**

  If $|\text{UB} - \text{LB}| \leq \epsilon$, then the optimal solution with a level of accuracy ϵ is obtained, otherwise $i \leftarrow i + 1$

- **Step 3: Solve master problem:** obtain the updated $x^{(i)}$ and the value of $\alpha^{(i)}$ as the updated LB, and go Step 1 with the updated fixed x

A more compact form of Benders' decomposition:

Subproblem:

$$\underset{x^{(i)},\,y^{(i)}}{\text{minimize}} \; -y^{(i)}$$

subject to

$$y^{(i)} - x^{(i)} \leq 5$$

$$y^{(i)} - \frac{x^{(i)}}{2} \leq \frac{15}{2}$$

$$y^{(i)} + \frac{x^{(i)}}{2} \leq \frac{35}{2}$$

$$y^{(i)} + x^{(i)} \leq 10$$

$$y^{(i)} \geq 0$$

$$x^{(i)} = x^{\text{fixed}\,(i)} \quad : \rho^{(i)}$$

Master problem:

$$\underset{x^{(i)},\,\alpha^{(i)}}{\text{minimize}} \; -\frac{x^{(i)}}{4} + \alpha^{(i)}$$

subject to

$$0 \leq x^{(i)} \leq 16$$

$$\alpha^{(i)} \geq \alpha^{\text{down}}$$

$$\alpha^{(i)} \geq -y^{(k)} + \rho^{(k)}\left[x^{(i)} - x^{(k)}\right] \quad \forall k = 1,..,i-1$$

Exercise:

The printed version of GAMS code of Benders' decomposition example is available on your table. Please check it in the next 10 minutes, then explain it to your neighbors around the table.

My colleagues and I will answer your questions!

This code has been prepared by Lejla Halilbasic, Christos Ordoudis, and Jalal Kazempour.

Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

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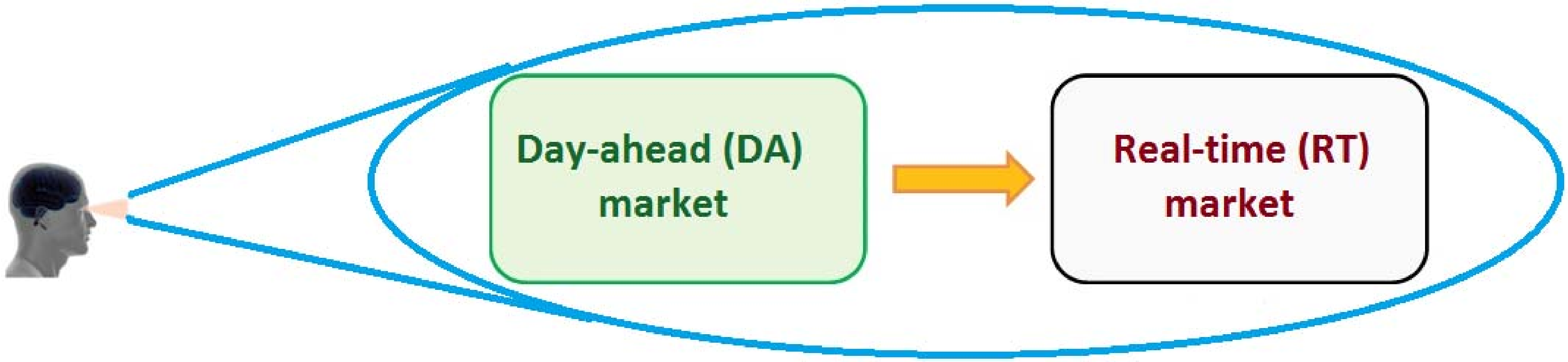
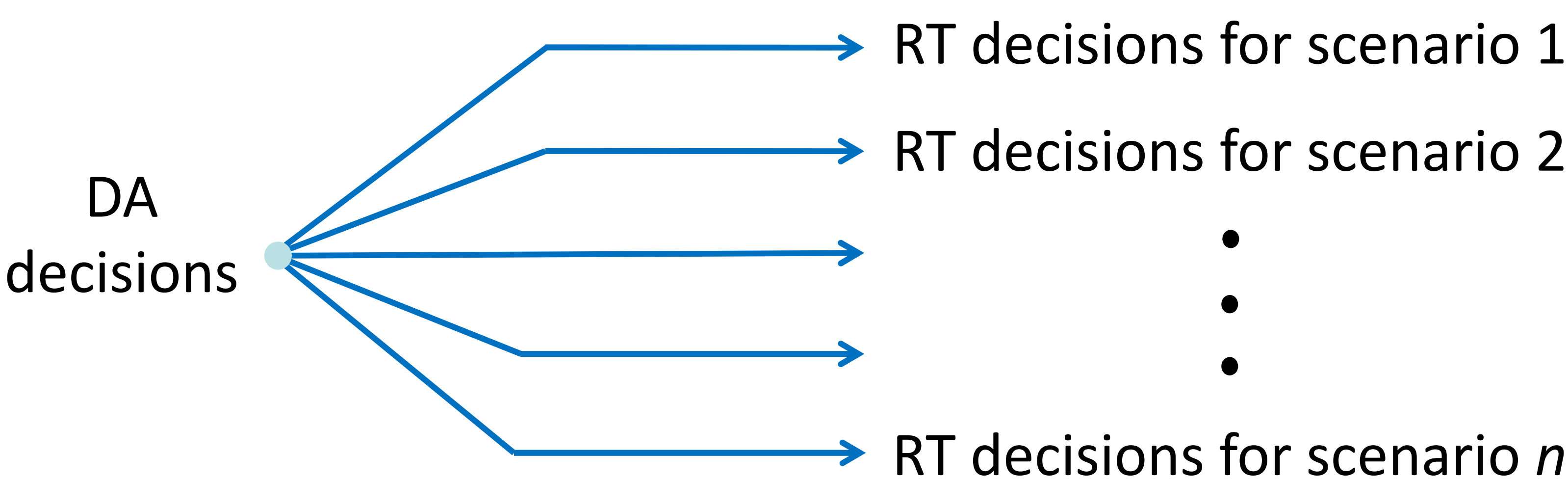
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem





Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

Day-ahead (DA) decisions:

Real-time (RT) decisions:

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Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

DTU

Day-ahead (DA) decisions:

- Power schedule [MW] of each generator ($\forall g: 1, .., G$), which is P_g.

Note: this variable is **scenario-independent**, in the sense that it should be adapted to all foreseen scenarios (here-and-now decisions).

Real-time (RT) decisions:

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Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

DTU

Day-ahead (DA) decisions:

- Power schedule [MW] of each generator ($\forall g: 1, .., G$), which is P_g.

Note: this variable is **scenario-independent**, in the sense that it should be adapted to all foreseen scenarios (here-and-now decisions).

Real-time (RT) decisions:

(For given DA decisions)

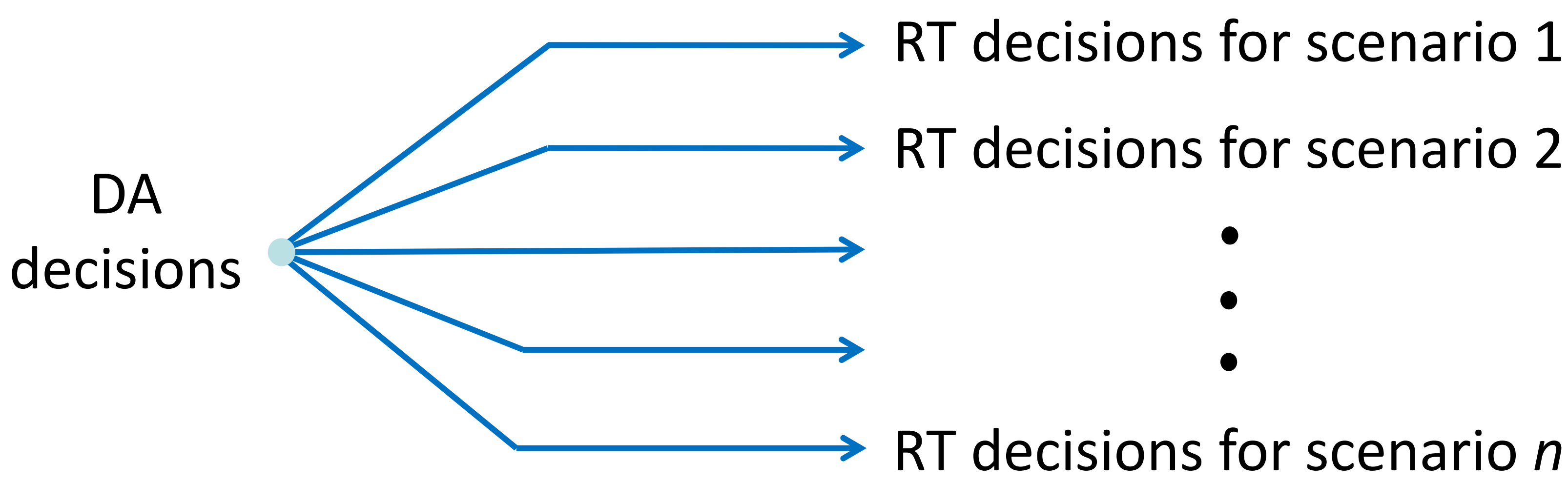
- Reserve deployment [MW] of each generator g under each foreseen scenario ($\forall s: 1, .., S$), which is $r_{g,s}$.
- Load shedding, wind curtailment, etc (all indexed by s).

Note: this variable is **scenario-dependent** (wait-and-see decisions).

Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

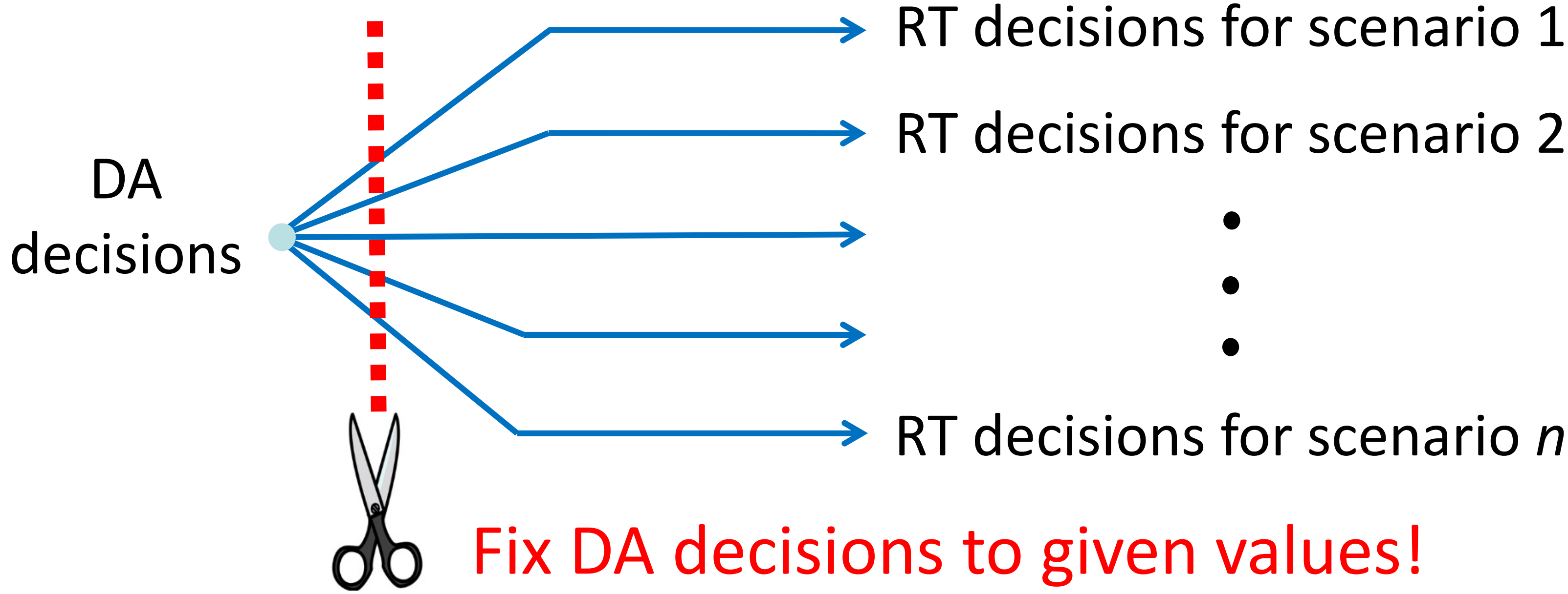
- Is there any <u>complicating variable</u> in this two-stage stochastic problem?



Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

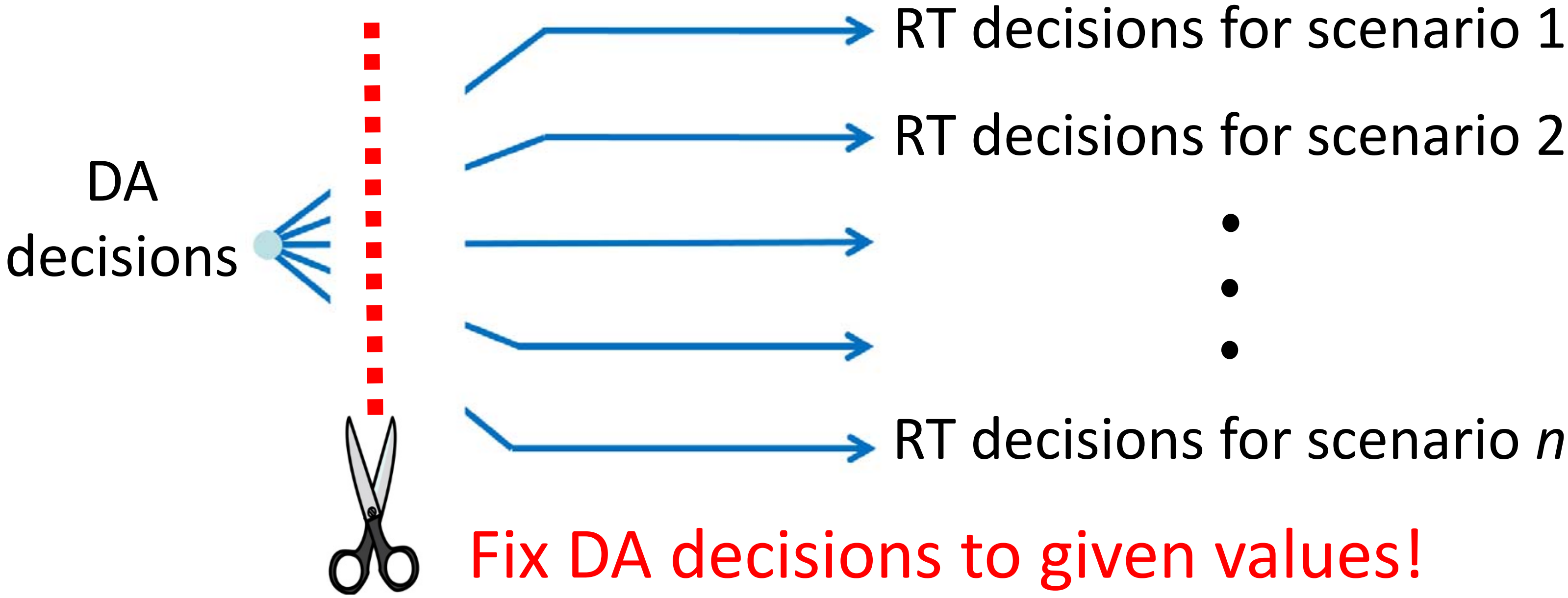
- Is there any complicating variable in this two-stage stochastic problem?



Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

- Is there any <u>complicating variable</u> in this two-stage stochastic problem?



Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

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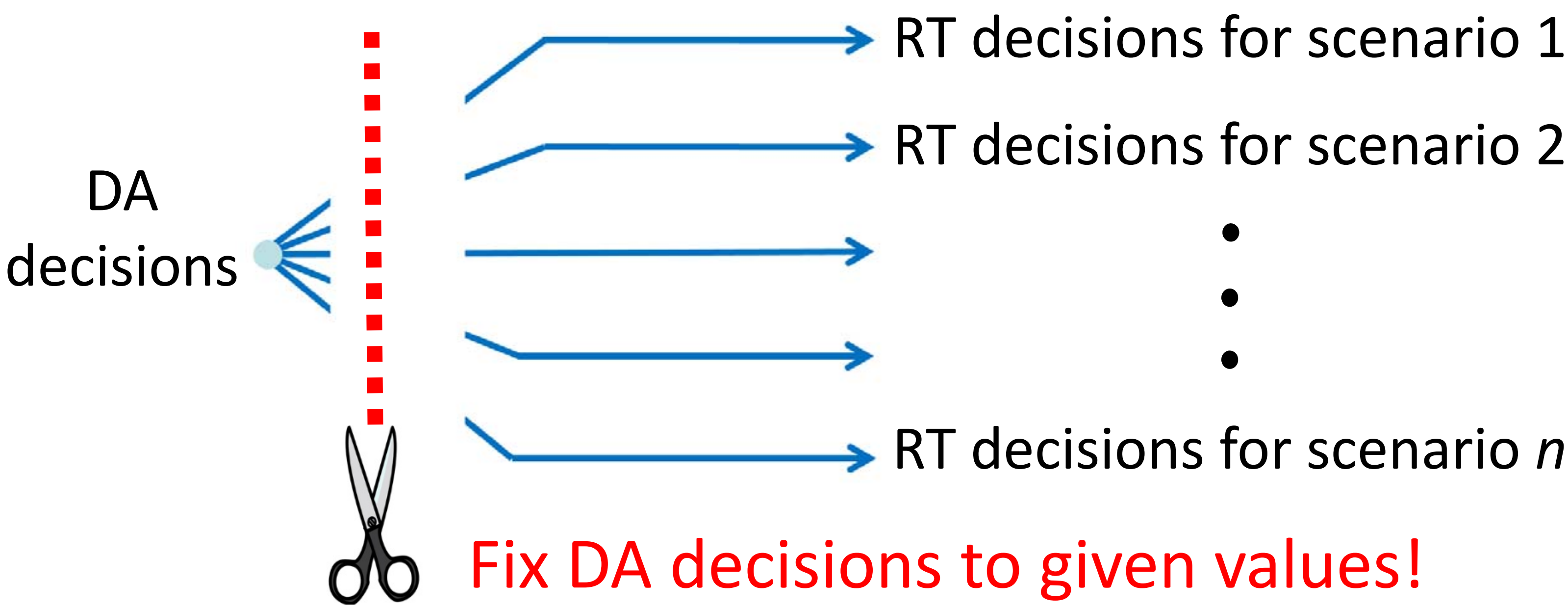


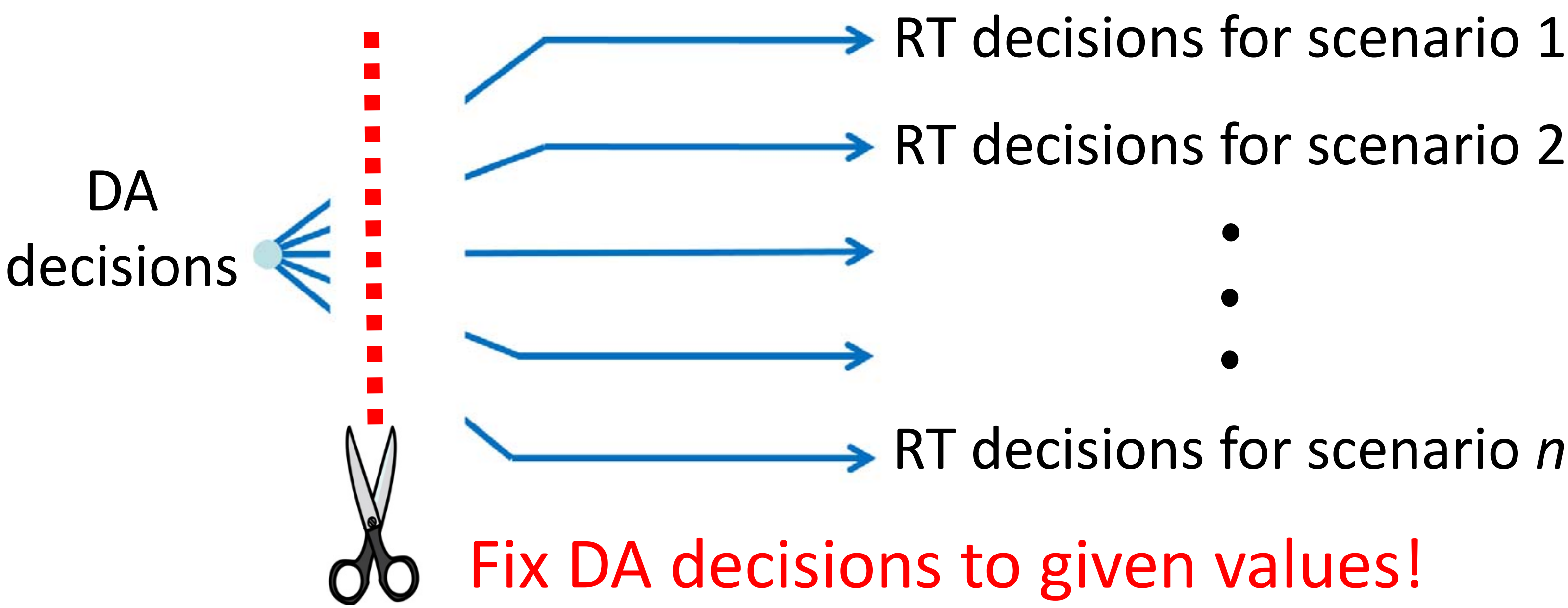
Fix DA decisions to given values!

Then, how many subproblems will you have?

Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

- Is there any <u>complicating variable</u> in this two-stage stochastic problem?

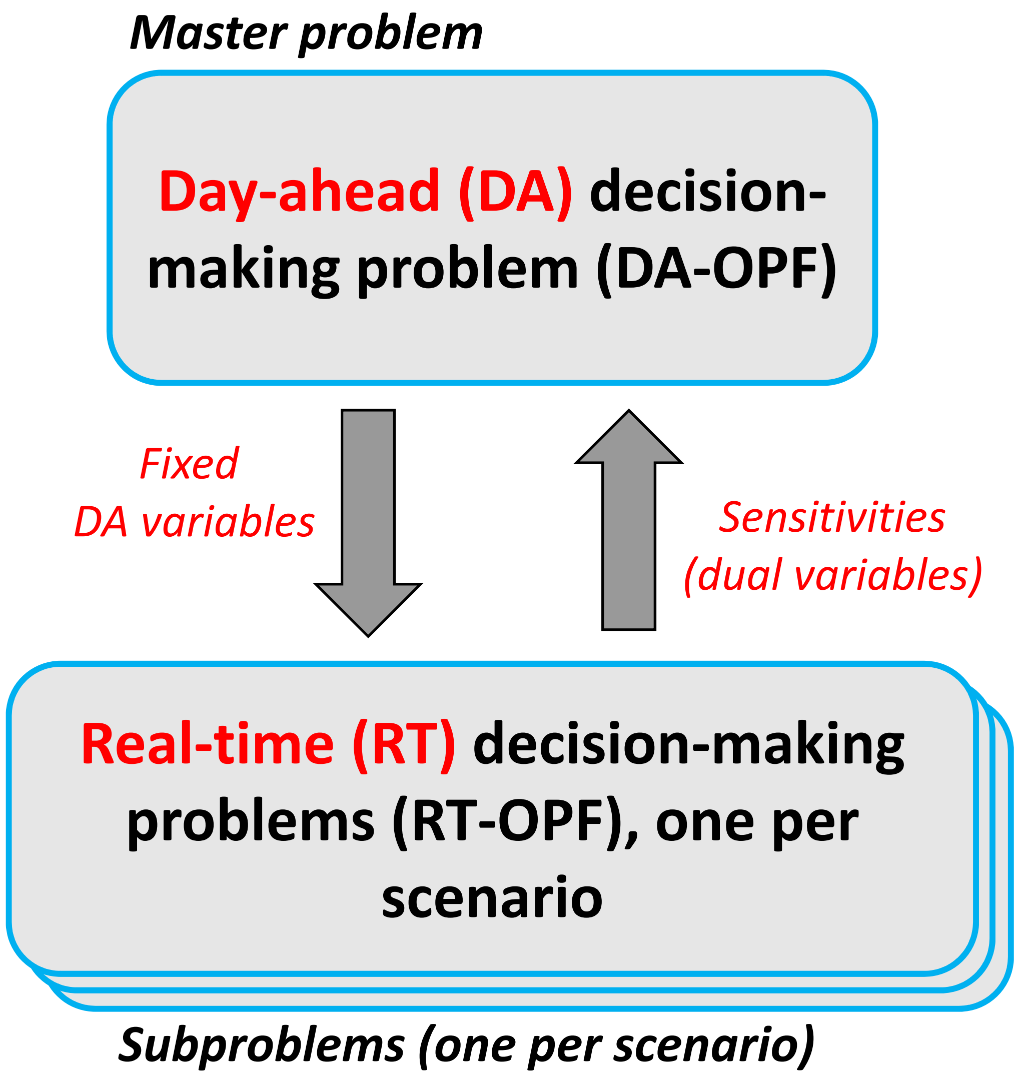


Fix DA decisions to given values!

Then, how many subproblems will you have? **One per scenario!**

Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem



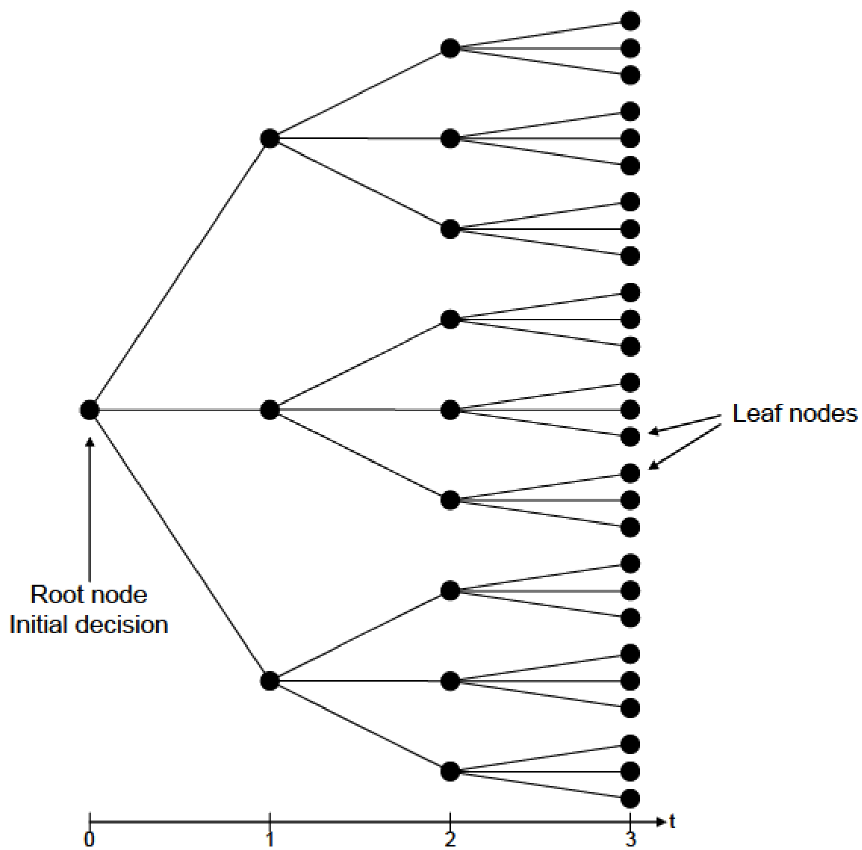
Two-stage Stochastic Problem

Two-stage (day-ahead and real-time) stochastic OPF problem

The application of Benders' decomposition to two-stage stochastic problems is also referred in the literature as **L-shaped decomposition**!

Some more thoughts

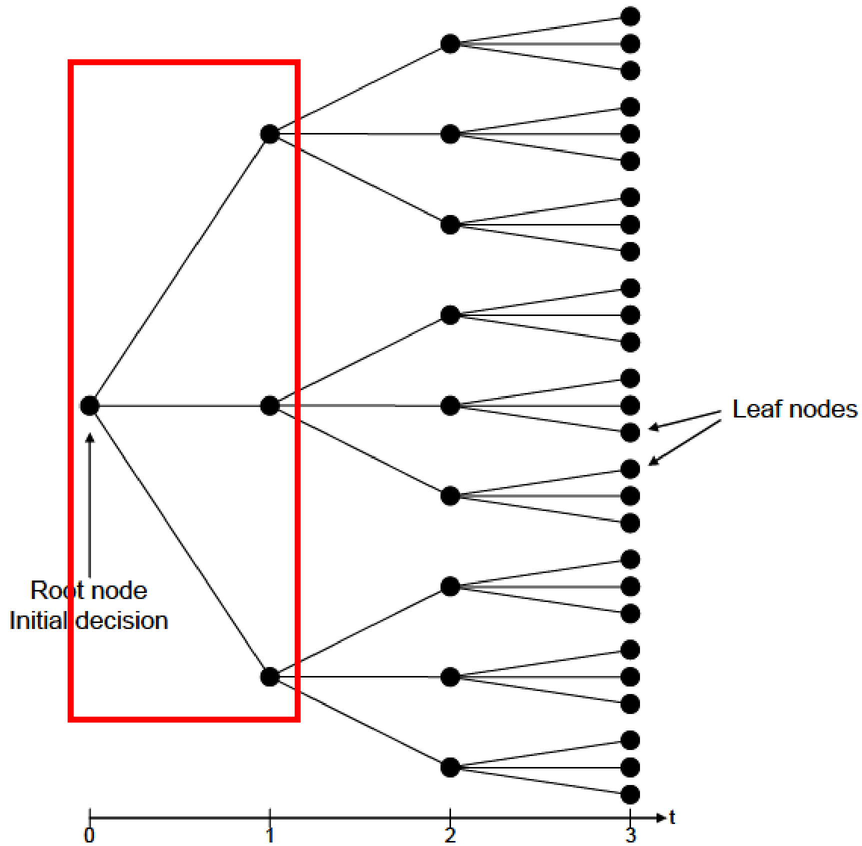
What to do in case of multi-stage (e.g., 4-stage) stochastic problem?



Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

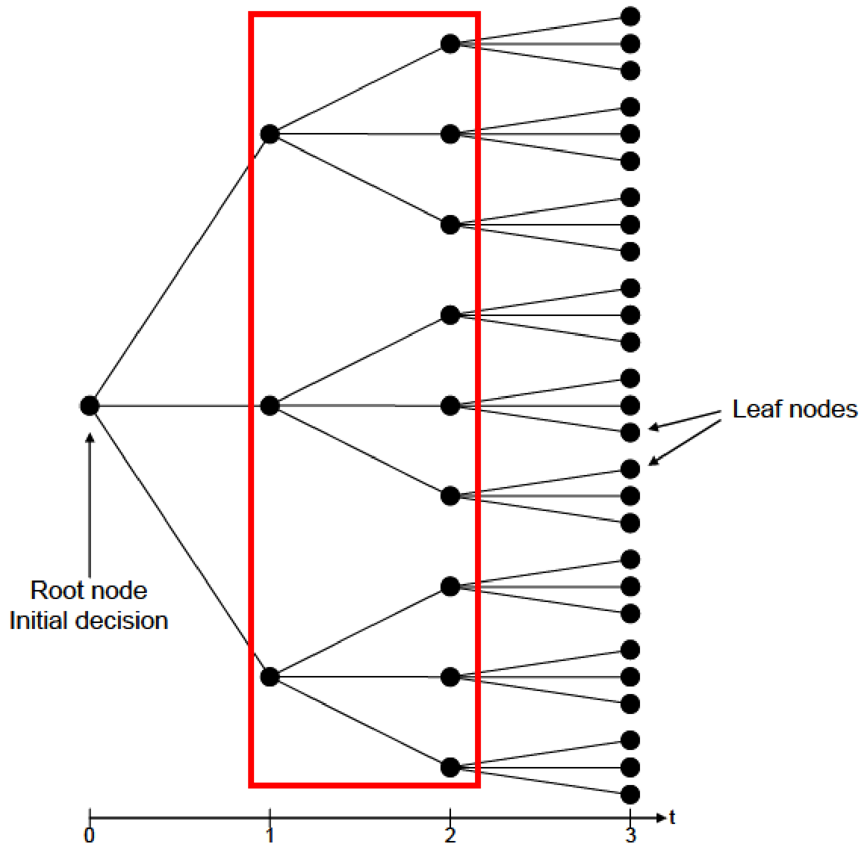
- **Step 1 in each iteration:** Solve a single two-stage stochastic problem
- **Master problem**: Stage 1
- **Subproblems:** Stage 2



Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

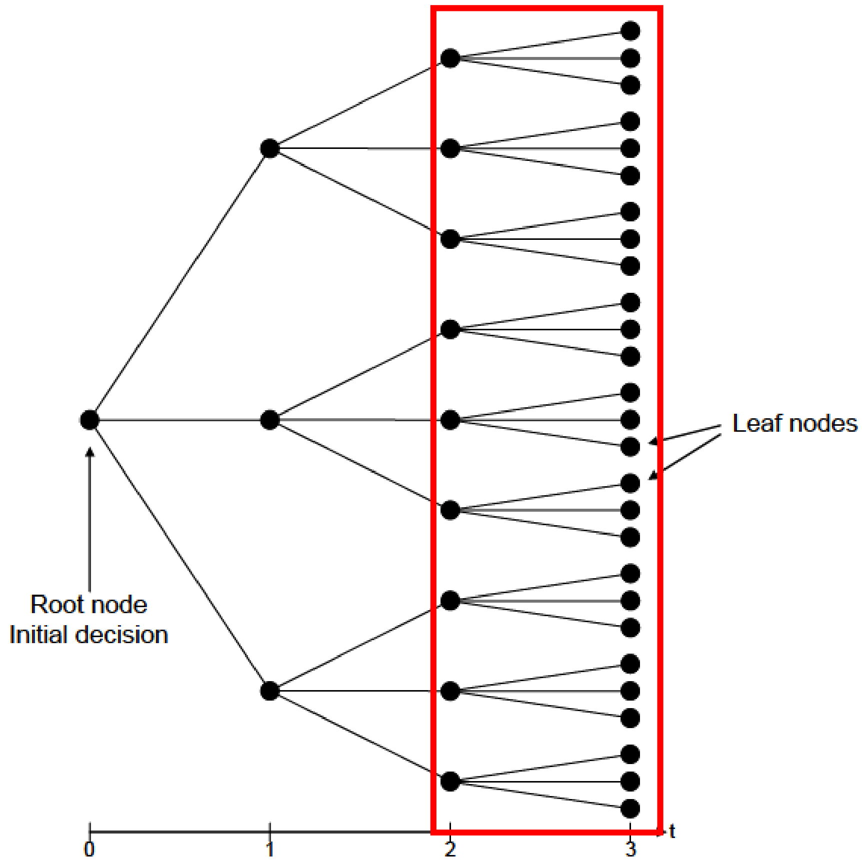
- **Step 2 in each iteration:** Solve 3 two-stage stochastic problems (separately)
- **Master problems**: Stage 2
- **Subproblems:** Stage 3



Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

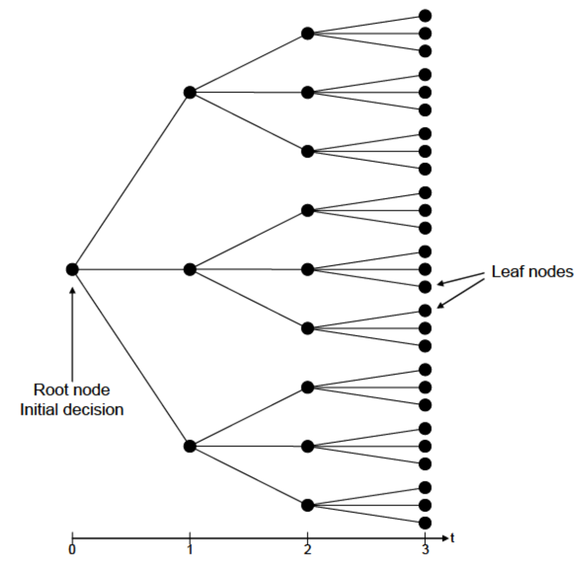
- **Step 3 in each iteration:** Solve 9 two-stage stochastic problems (separately)
- **Master problems**: Stage 3
- **Subproblems:** Stage 4



Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

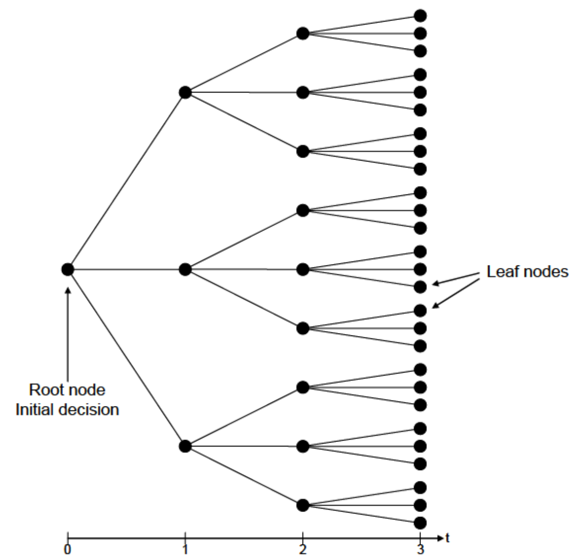
Each multi-stage stochastic problem is a collection of **nested** two-stage stochastic problems



Some more thoughts

What to do in case of multi-stage (e.g., 4-stage) stochastic problem?

Each multi-stage stochastic problem is a collection of **nested** two-stage stochastic problems



Techniques to be used:

- Nested Benders' decomposition
- Stochastic dual dynamic programming (SDDP)

Reference: J. Murphy, "Benders, Nested Benders and Stochastic Programming: An Intuitive Introduction", *Cambridge University Engineering Department Technical Report*, 2013.

Some more thoughts

What to do in a case in which subproblems include **binary (0/1)** variables?

Some more thoughts

What to do in a case in which subproblems include **binary (0/1)** variables?

The trouble?

Sensitivities (dual variables) cannot be obtained in a discrete feasible region! But, we need those sensitivities to generate cuts in the master problem!

Some more thoughts

What to do in a case in which subproblems include **binary (0/1)** variables?

Solution:

- Generate cuts in the master problem based on the values obtained for **"primal" variables, and not "dual" variables** of the subproblems!

Some more thoughts

What to do in a case in which subproblems include **binary (0/1)** variables?

Solution technique:

- Primal Benders' decomposition (cutting-plane method)

  This technique has recently been used in power systems applications for two-stage robust optimization problems

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Thanks for your attention!

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